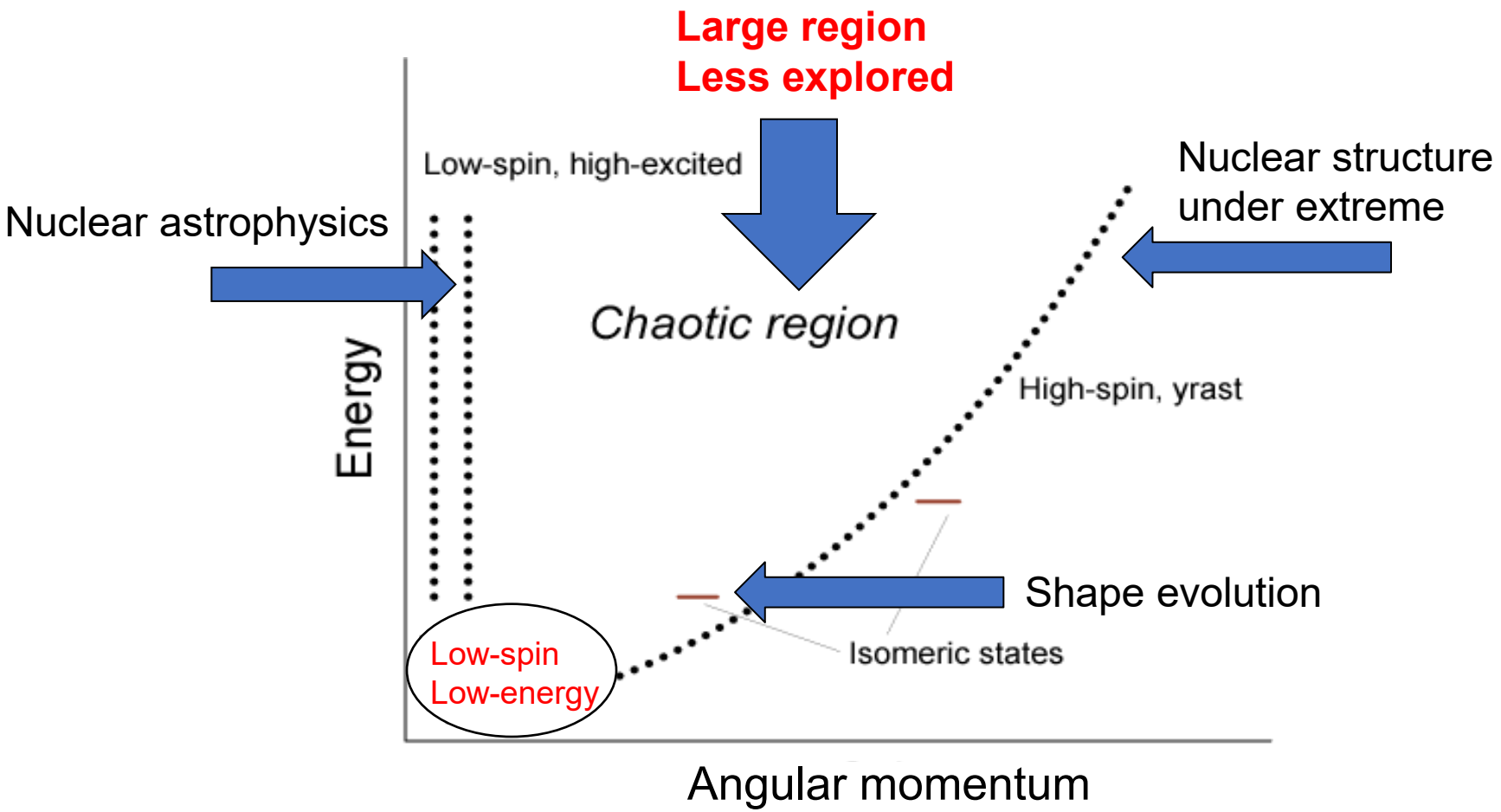
Structural Effects in Nuclear Statistics — A Microscopic View of Highly Excited States

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Nuclear excitations



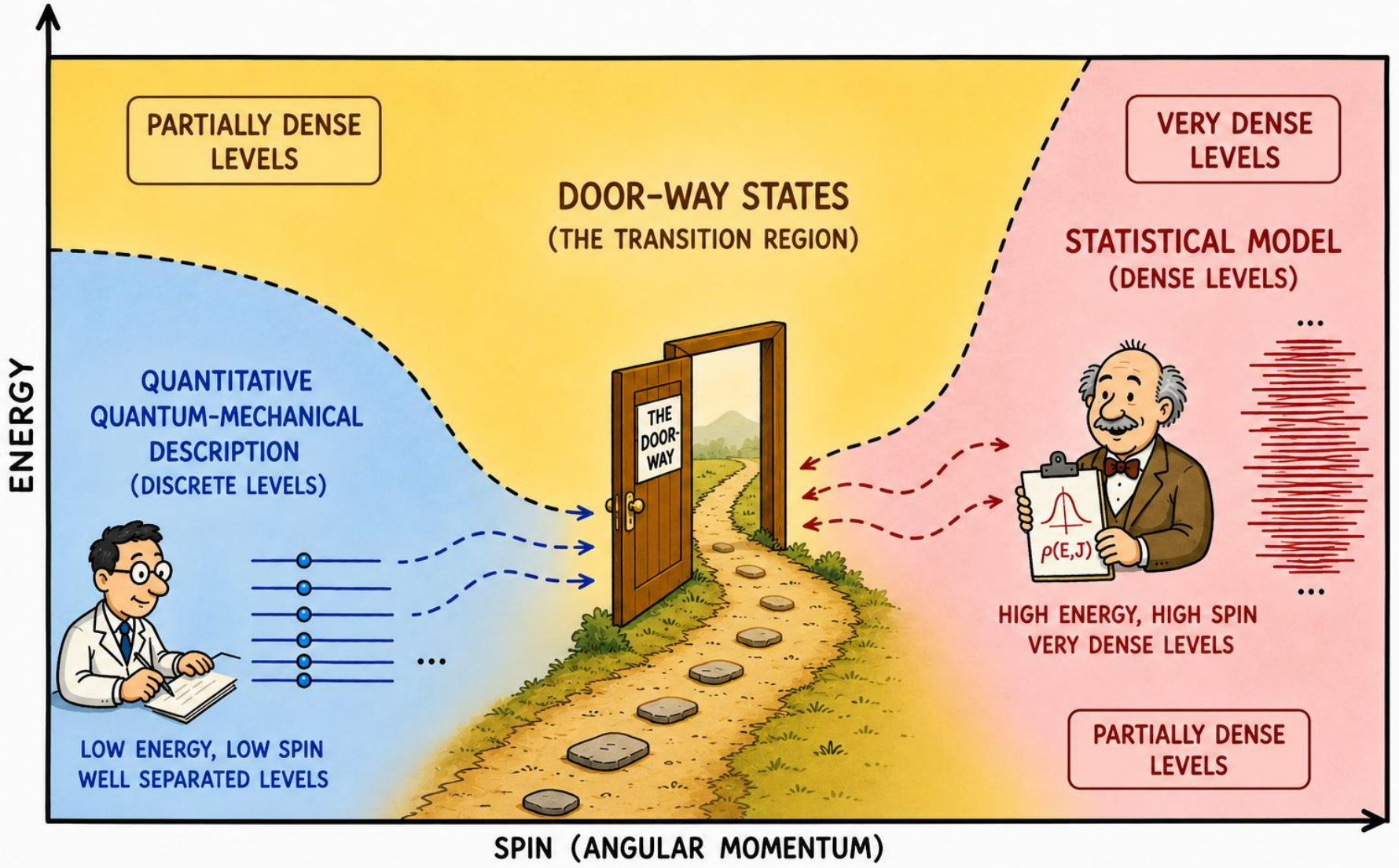
The standard view of the **quasi-continuum** is that the nucleus behaves like a many-body quantum **chaotic** system, where individual states lose distinct structure completely.

However, growing number of evidence suggests this may not be the case!

V. Zelevinsky, B. A. Brown, N. Frazier, and M. Horoi, Phys. Rep. 276, 85 (1996)

H. A. Weidenmüller and G. E. Mitchell, Rev. Mod. Phys. 81, 539 (2009)

The door way states



The **quasi-continuum** may be the place to find **door-way** states.

There, levels are **dense enough** to be resolved, yet they are not completely structureless but carry **structure features.**

ChatGPT created image

How to study the door-way states?

- Growing number of evidence that calls for study of door-way states

  Y. Sun, *Are highly excited nuclear states merely chaotic?*
  to appear in Proceedings of INPC-2025.

- **Two directions** to connect the two extreme regions by the door-way states:

- **Start from a shell model** that uses **physics-dictated algorithm** to construct a large many-body configuration space, and generate eigenstates to build level density and γ-ray strength functions.

  Y. Sun, *Projection techniques to approach the nuclear many-body problem,*
  Phys. Scr. 91, 043005 (2016).

- **Start from a statistical model**, and try to put shell-structure information in it
  → **structure-dictated statistics.**

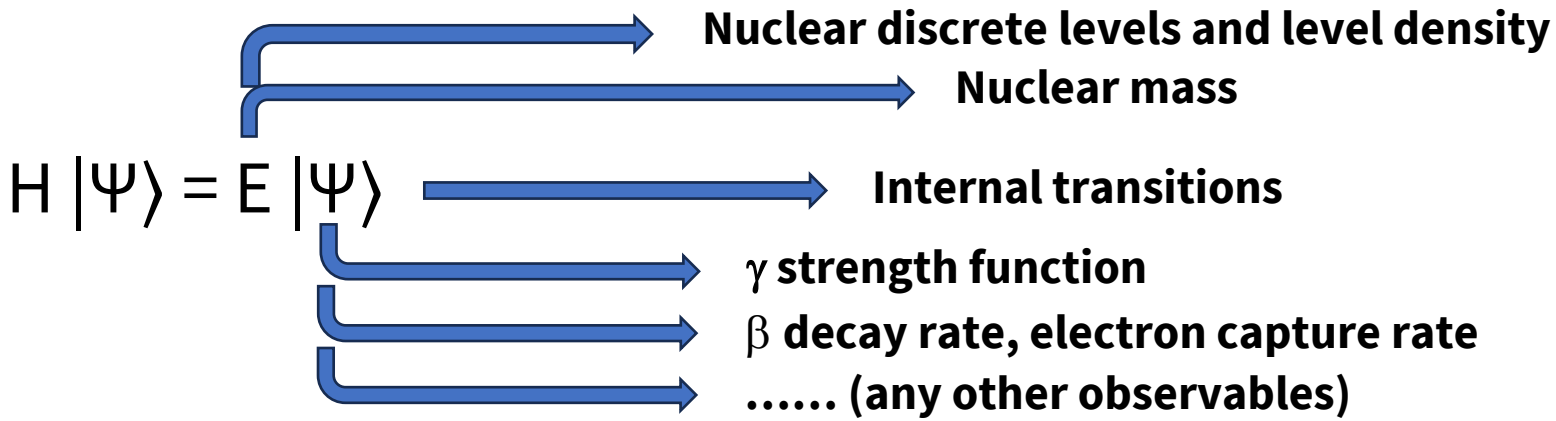
  J.-C. Guo and Y. Sun, *Symmetry-imposed correlation in nuclear level statistics: The spin distribution*,
  to be published.

A “complete set” of nuclear levels

- A complete set:
  - Begin from the ground state and extend the calculation **to and beyond** the neutron separation energy, with **good spin and parity** assignments to each level.
- Experiments to collect these levels:
  - Inelastic scattering, pickup, and transfer reactions; γ-ray spectroscopy following β or α decay; heavy-ion excitation; etc.
  - Neutron capture reaction is a powerful method.
- These experiments are quite selective. One needs to combine different approaches to ensure that all levels (for given excitations) are observed.
- On the theory side, the whole problem of a microscopic study is on the crossing of **many-body quantum physics, statistical method of small isolated systems, and quantum chaos.** **(Zelevinsky)**
- **It is a difficult problem, both experimentally and theoretically.**

Calculation of highly excited states

- Ideally, the levels are obtained by solving the eigenvalue equation. Once the equation is solved, one can calculate **everything**.



- **But,** can we solve this question to obtain complete set of levels with good spin/parity for arbitrarily heavy, deformed nuclei?
- **Clearly, this is the task that cannot be performed by any traditional models.**

Projected shell model: A shell model with broken pair configurations

- Most nuclei in the nuclear chart are, more or less, **deformed.**
- One can start from a **deformed basis** where the rotational symmetry is broken spontaneously.

  Y. Sun, Phys. Scr. 91 (2016) 043005

- In deformed bases, intrinsic states are classified in **pair-breaking configurations.**
- One then applies **angular-momentum-projection** technique to recover the broken symmetry, which transforms intrinsic states to the laboratory frame.

  M. Guidry and Y. Sun, *Symmetry, Breaking Symmetry, and Topology in Modern physics* (Cambridge University Press, 2022)

- One finally diagonalizes the Hamiltonian in the (angular-momentum) projected basis, similar as a conventional shell model.
- This is the basic concept of the **Projected Shell Model.**

  K. Hara, Y. Sun, Int. J. Mod. Phys. E4 (1995) 637

How angular momentum can be calculated?

- How angular momentum of many-body states is calculated in theory?
- Two methods: **angular momentum coupling** and **angular momentum projection**
- **Angular momentum coupling:**
  - **J- scheme** uses mathematical method (Coefficients of Fractional Parentage, CFP). With increasing number of active nucleons, this method becomes extremely complex in practical applications.
  - **M-scheme** allows admixture of J with $J \geq |M|$, so the many-body basis is no longer eigenstates of angular momentum. The resulting basis is huge – therefore the name of **large-scale shell model**.

    A. de-Shalit and I. Talmi, Nuclear Shell Theory (Academic Press, New York and London, 1963).
- **Angular momentum projection:**
  - **on deformed basis** to recover violated angular momentum by deformation.

    P. Ring and P. Schuck, The Nuclear Many-Body Problem (Springer–Verlag, New York, 1980).
  - **on spherical basis**, new efficient method for angular-momentum coupling.

    J.-C. Guo and Y. Sun, Phys. Rev. C 112, 064307 (2025).

A novel algorithm for shell-model calculation

Hypothesis:

A “complete” shell-model basis can be constructed by

Projected Shell Model with

pair-breaking states

(a natural hierarchy in configurations)

There are experimental evidences to support this hypothesis.

Y. Sun, Phys. Scr. 91 (2016) 043005

1. Experimental evidence of pair breaking

- **Experimental evidence:** Moment of inertia (MoI) along the Yrast line changes violently with rotation.

$$\mathcal{J}(I) = \frac{2I-1}{E(I)-E(I-2)}$$

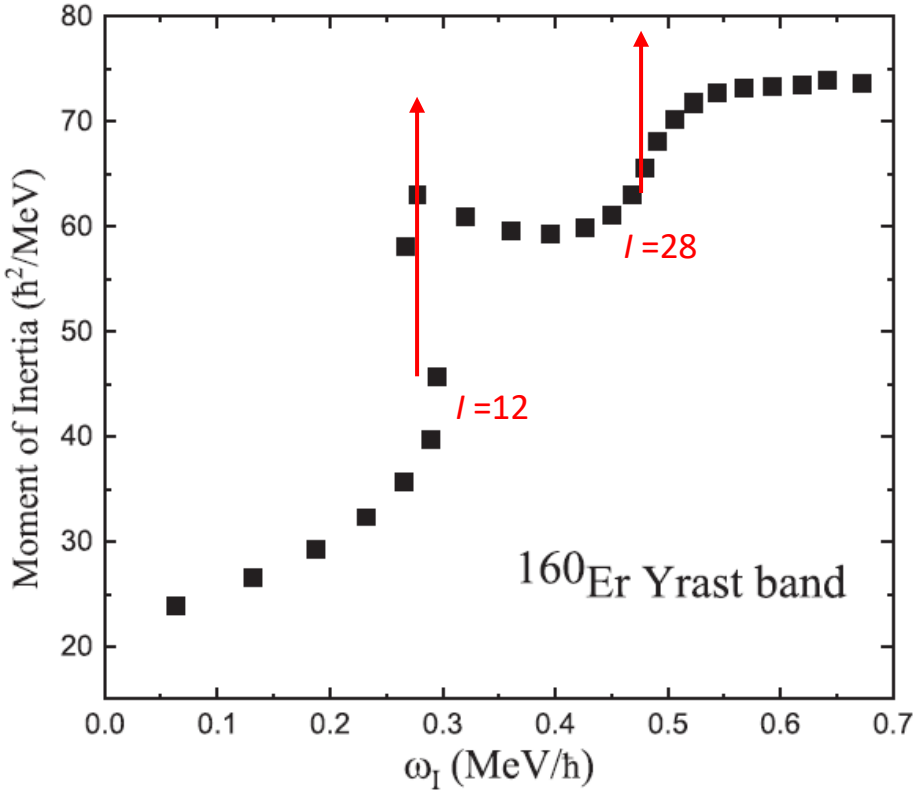
- **The Stephens-Simon Effect:**
- At *I* ~ 12, MoI suddenly increases. This is the effect of first nucleon-pair breaking → spin alignment

F. S. Stephens and R. S. Simon, Nucl. Phys. A 183, 257 (1972)

- At *I* ~ 28, an additional nucleon pair is broken simultaneously. MoI increases again.

I. Y. Lee et al. Phys. Rev. Lett. 38, 1454 (1977)

- **This hints us to include 0-qp, 2-qp, 4-qp configurations in the shell model calculation.**

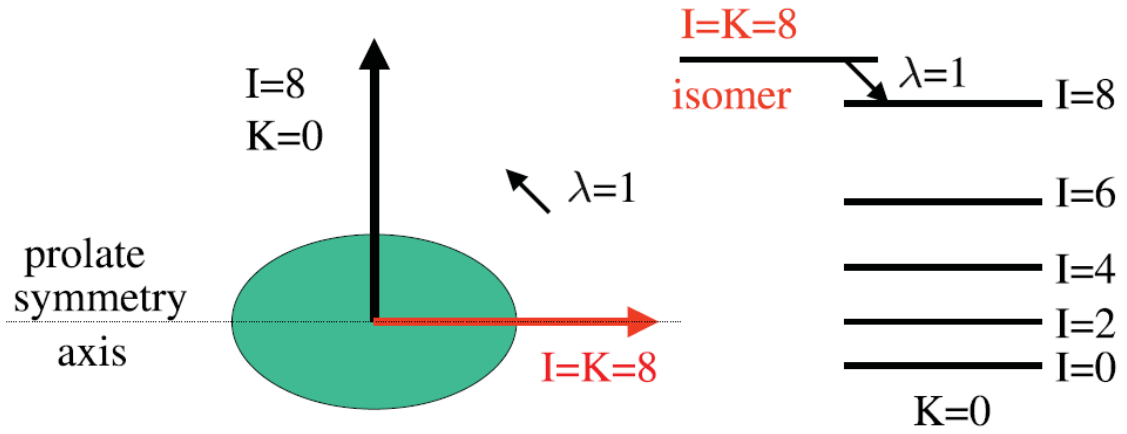


K. Hara and Y. Sun, Int. J. Mod. Phys. E4 (1995) 637

2. Occurrence of multi-qp *K*-isomers

- **Experimental evidence:** the structure of high-*K* isomers is built from **broken nucleon pairs** that generate **high *K* states** aligned with nuclear symmetry axis.
- There are evidences for 6- and 8-qp high-*K* isomers.
- **Each of the isomers correspond to a 6- or 8- qp state.**

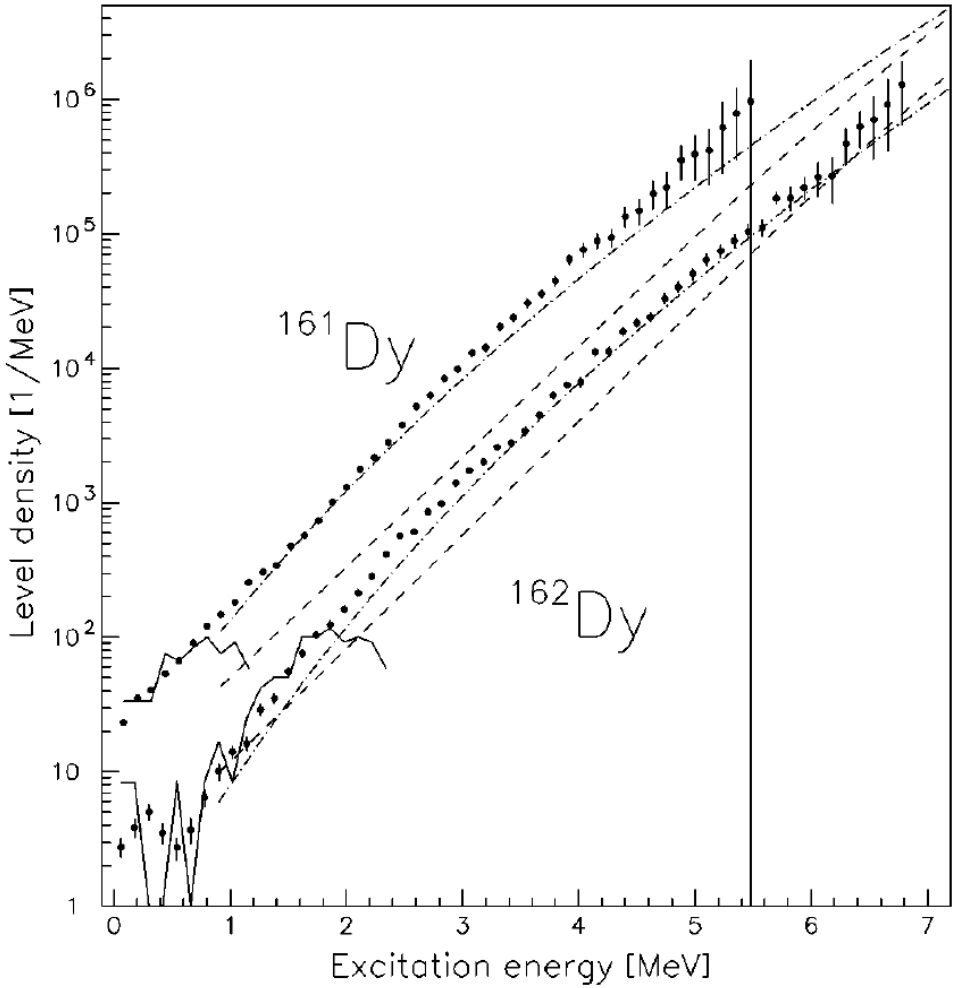
C. S. Purry, et al., Phys. Rev. Lett. 75, 406 (1995).
G. D. Dracoulis, et al., Phys. Rev. C 71, 044326 (2005).



P. M. Walker, Phys. Scr. 92, 054001 (2017)
G. D. Dracoulis, P. M. Walker, F. G. Kondev, Rep. Prog. Phys. 79, 076301 (2016)

3. Increasing entropy in odd-mass systems

- **Experimental evidence:** For a given excitation, level density of odd-mass nuclei increases by a pairing gap Δ.
- Guttormsen et al. examined 280 nuclei and suggested it as the odd-even entropy differences.
- This can be understood as in odd-mass nuclei, the number of one last nucleon is blocked from the pair formation.
- **This effect is naturally included in our pair-breaking configuration space:**
  even-even (0-qp, 2-qp, 4-qp, …);
  odd-mass (1-qp, 3-qp, 5-qp …);
  odd-odd (2-qp, 4-qp, 6-qp, …).



M. Guttormsen, et al., Phys. Rev. C 61, 067302 (2000); Phys. Rev. C 62, 024306 (2000).

Multi-qp configurations in even-even nuclei

- **Many-body configuration space** consists of such **quasi-particle (qp) states**, for example: 0-, 2-, 4-, 6, 8, up to 10-qp states for even-even nuclei.
- Indices cover all possible single-particles subject to the Pauli Principle.

0-qp: 1st hierarchy

2-qp: 2nd hierarchy

4-qp: 3rd hierarchy

6-qp: 4th hierarchy

8-qp: 5th hierarchy

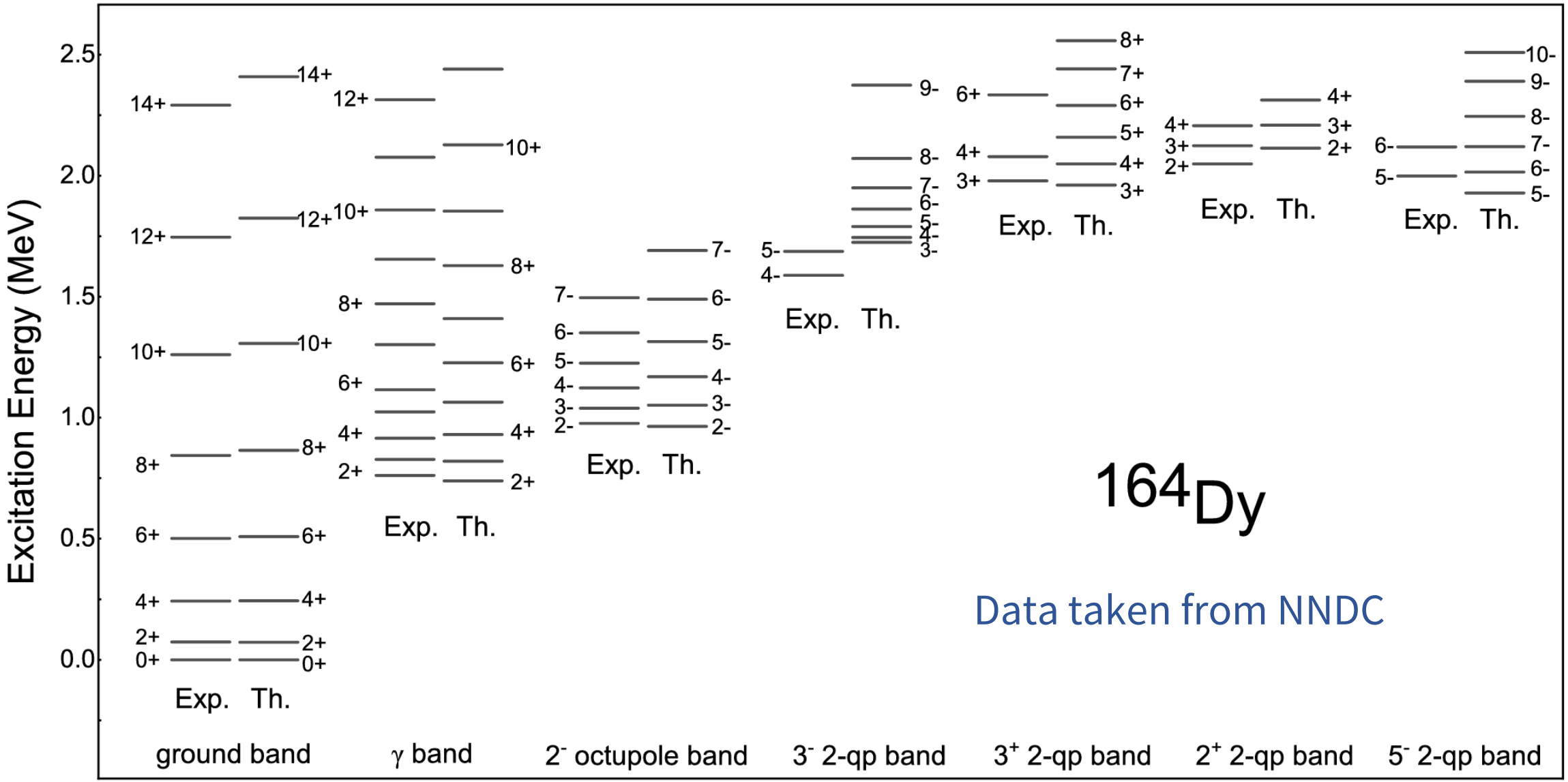
10-qp: 6th hierarchy

$$\Big\{ |\Phi\rangle,\ a^\dagger_{\nu_i}a^\dagger_{\nu_j}|\Phi\rangle,\ a^\dagger_{\pi_i}a^\dagger_{\pi_j}|\Phi\rangle,\ a^\dagger_{\nu_i}a^\dagger_{\nu_j}a^\dagger_{\pi_k}a^\dagger_{\pi_l}|\Phi\rangle,$$
$$a^\dagger_{\nu_i}a^\dagger_{\nu_j}a^\dagger_{\nu_k}a^\dagger_{\nu_l}|\Phi\rangle,\ a^\dagger_{\pi_i}a^\dagger_{\pi_j}a^\dagger_{\pi_k}a^\dagger_{\pi_l}|\Phi\rangle,$$
$$a^\dagger_{\nu_i}a^\dagger_{\nu_j}a^\dagger_{\nu_k}a^\dagger_{\nu_l}a^\dagger_{\nu_m}a^\dagger_{\nu_n}|\Phi\rangle,\ a^\dagger_{\pi_i}a^\dagger_{\pi_j}a^\dagger_{\pi_k}a^\dagger_{\pi_l}a^\dagger_{\pi_m}a^\dagger_{\pi_n}|\Phi\rangle,$$
$$a^\dagger_{\pi_i}a^\dagger_{\pi_j}a^\dagger_{\nu_k}a^\dagger_{\nu_l}a^\dagger_{\nu_m}a^\dagger_{\nu_n}|\Phi\rangle,\ a^\dagger_{\nu_i}a^\dagger_{\nu_j}a^\dagger_{\pi_k}a^\dagger_{\pi_l}a^\dagger_{\pi_m}a^\dagger_{\pi_n}|\Phi\rangle,$$
$$a^\dagger_{\pi_i}a^\dagger_{\pi_j}a^\dagger_{\nu_k}a^\dagger_{\nu_l}a^\dagger_{\nu_m}a^\dagger_{\nu_n}a^\dagger_{\nu_o}a^\dagger_{\nu_p}|\Phi\rangle, a^\dagger_{\pi_i}a^\dagger_{\pi_j}a^\dagger_{\pi_k}a^\dagger_{\pi_l}a^\dagger_{\nu_m}a^\dagger_{\nu_n}a^\dagger_{\nu_o}a^\dagger_{\nu_p}|\Phi\rangle,$$
$$a^\dagger_{\pi_i}a^\dagger_{\pi_j}a^\dagger_{\pi_k}a^\dagger_{\pi_l}a^\dagger_{\nu_m}a^\dagger_{\nu_n}a^\dagger_{\nu_o}a^\dagger_{\nu_p}a^\dagger_{\nu_q}a^\dagger_{\nu_r}|\Phi\rangle,$$
$$a^\dagger_{\nu_i}a^\dagger_{\nu_j}a^\dagger_{\nu_k}a^\dagger_{\nu_l}a^\dagger_{\pi_m}a^\dagger_{\pi_n}a^\dagger_{\pi_o}a^\dagger_{\pi_p}a^\dagger_{\pi_q}a^\dagger_{\pi_r}|\Phi\rangle\Big\} \qquad (1)$$

Each hierarchy starts by ~2 MeV higher than the previous one.

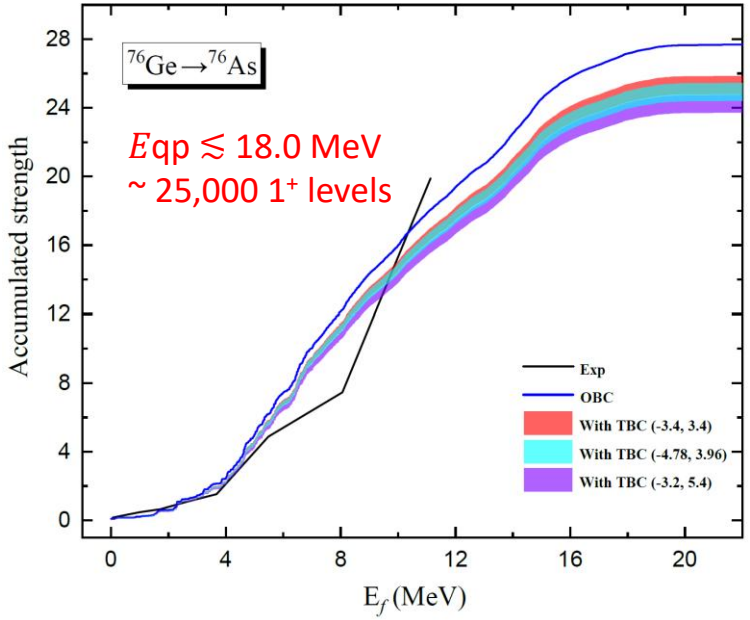
L.-J. Wang et al, Phys. Rev. C 90 (2014) 011303(R)
L.-J. Wang et al, Phys. Rev. C 93 (2016) 034322

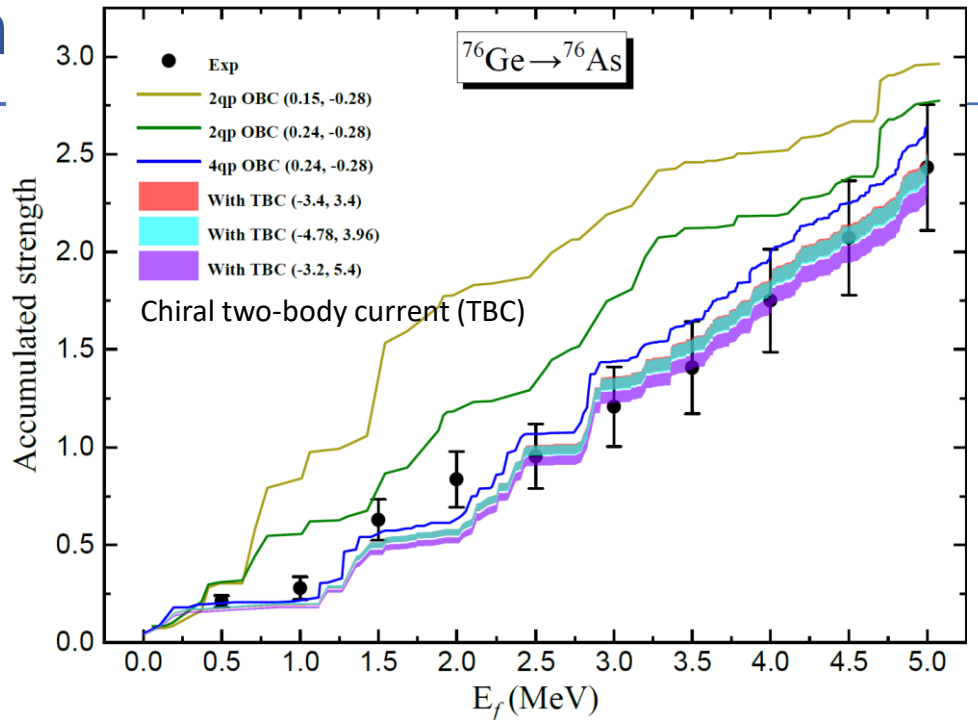
^{164}Dy: comparison with known discrete data

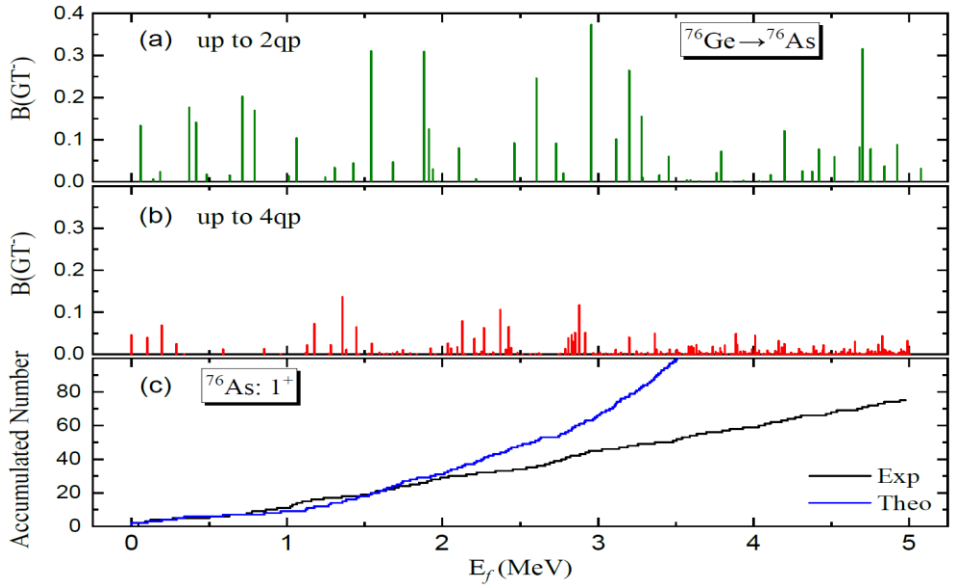


An example: ^{76}Ge→^{76}As β-decay calculation

- β-decay calculations **without applying a quenching factor**
- Data from charge-exchange reaction by J. H. Thies *et al.*, Phys. Rev. C 86 (2012) 014304.
- Up to 2qp: ~ 15 1^+ states in 4-5 MeV energy bin.
- Up to 4qp: ~ 150 1^+ states in 4-5 MeV energy bin.
- With more qp states, GT strength is **pushed to high energies**.
- **A broad enhancement in strength is seen from 5 - 15 MeV,** without prominent resonance peaks, consisting of thousands of individual transitions within each 1 MeV bin.

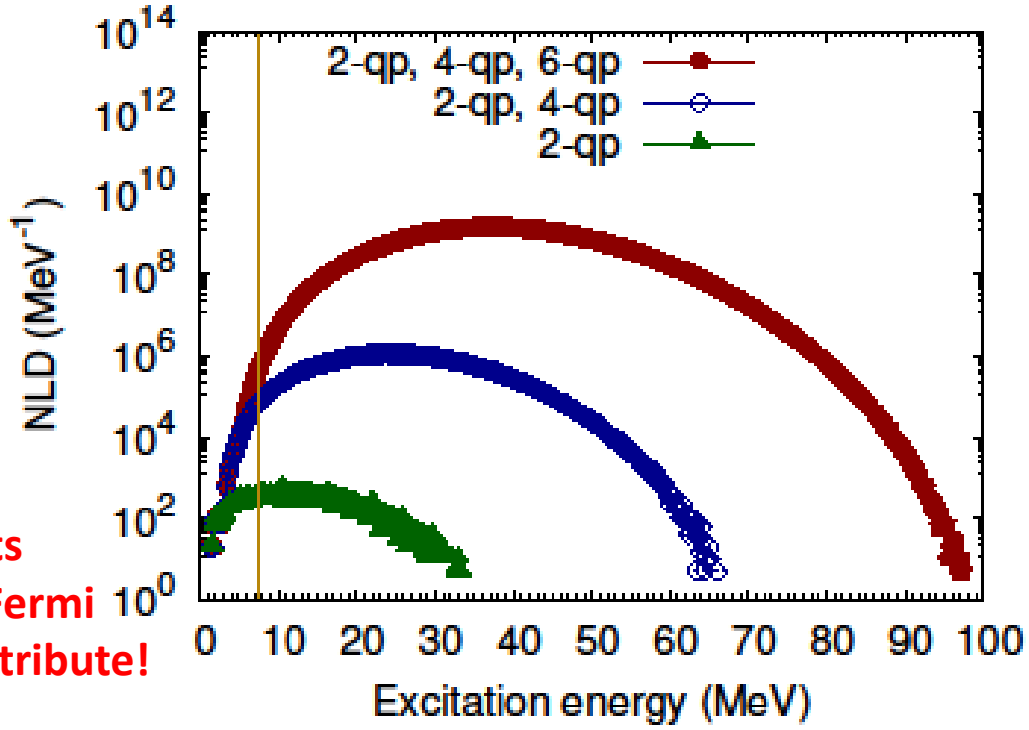






What configurations contribute to level density?

- Based on the algorithm of the projected shell model, we construct **all possible many-body states of spin and parity**, exhausting three major shells both for neutrons and protons.

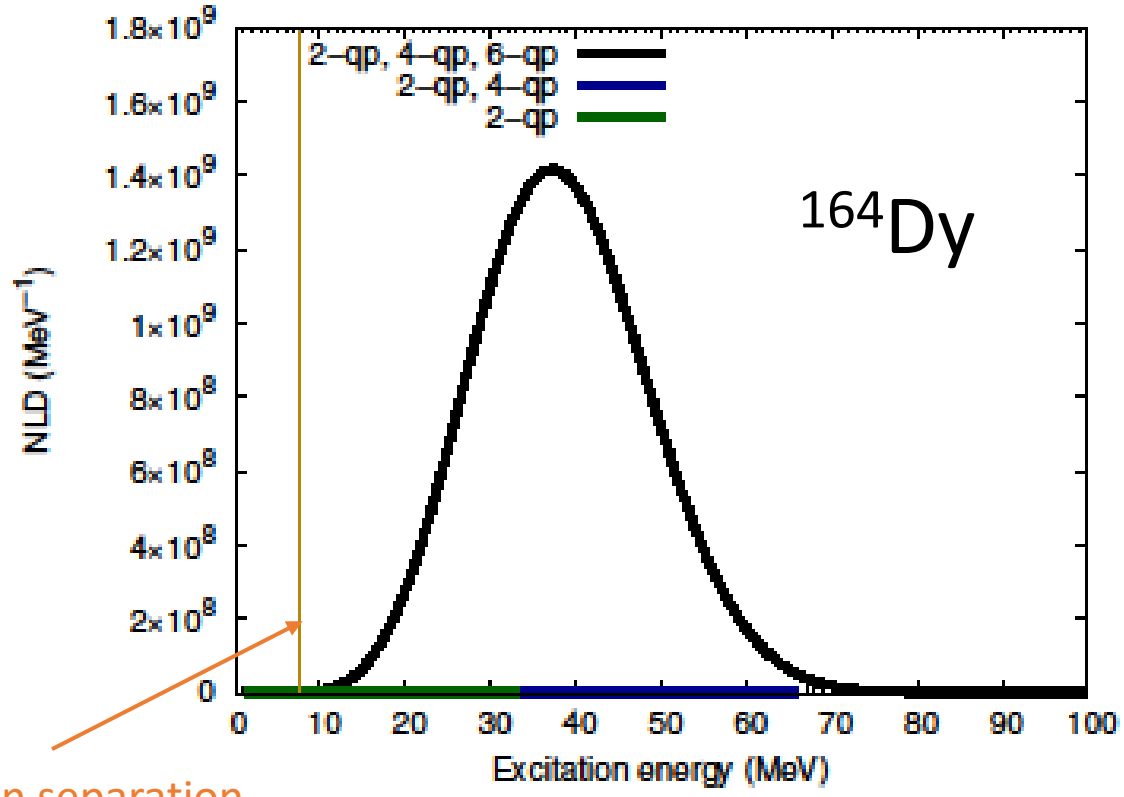


Only orbits near the Fermi levels contribute!

- Our usual level density occupies only a **tiny low-energy part** of the full configuration space.

Full set of qp configurations coupled from 6 Nilsson major shells

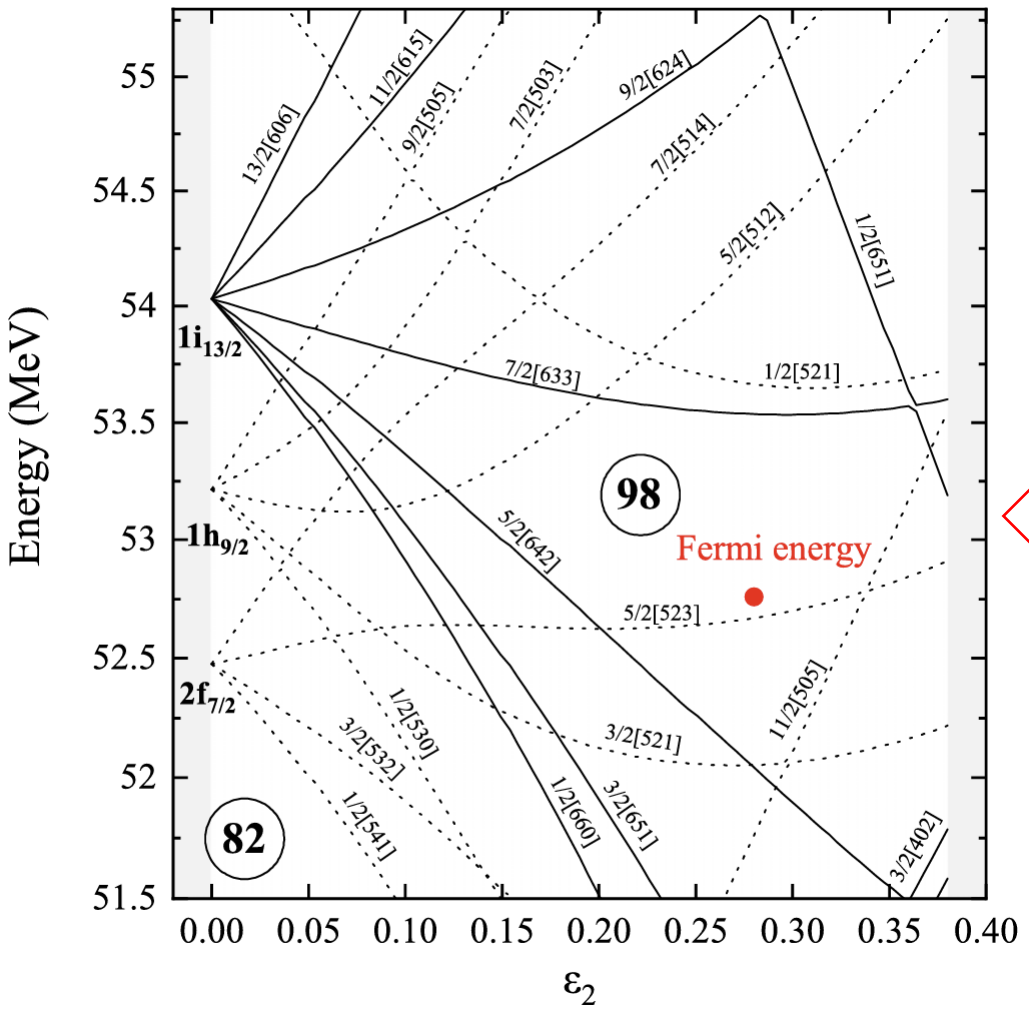
Neutrons: N=4,5,6, protons: N=3,4,5



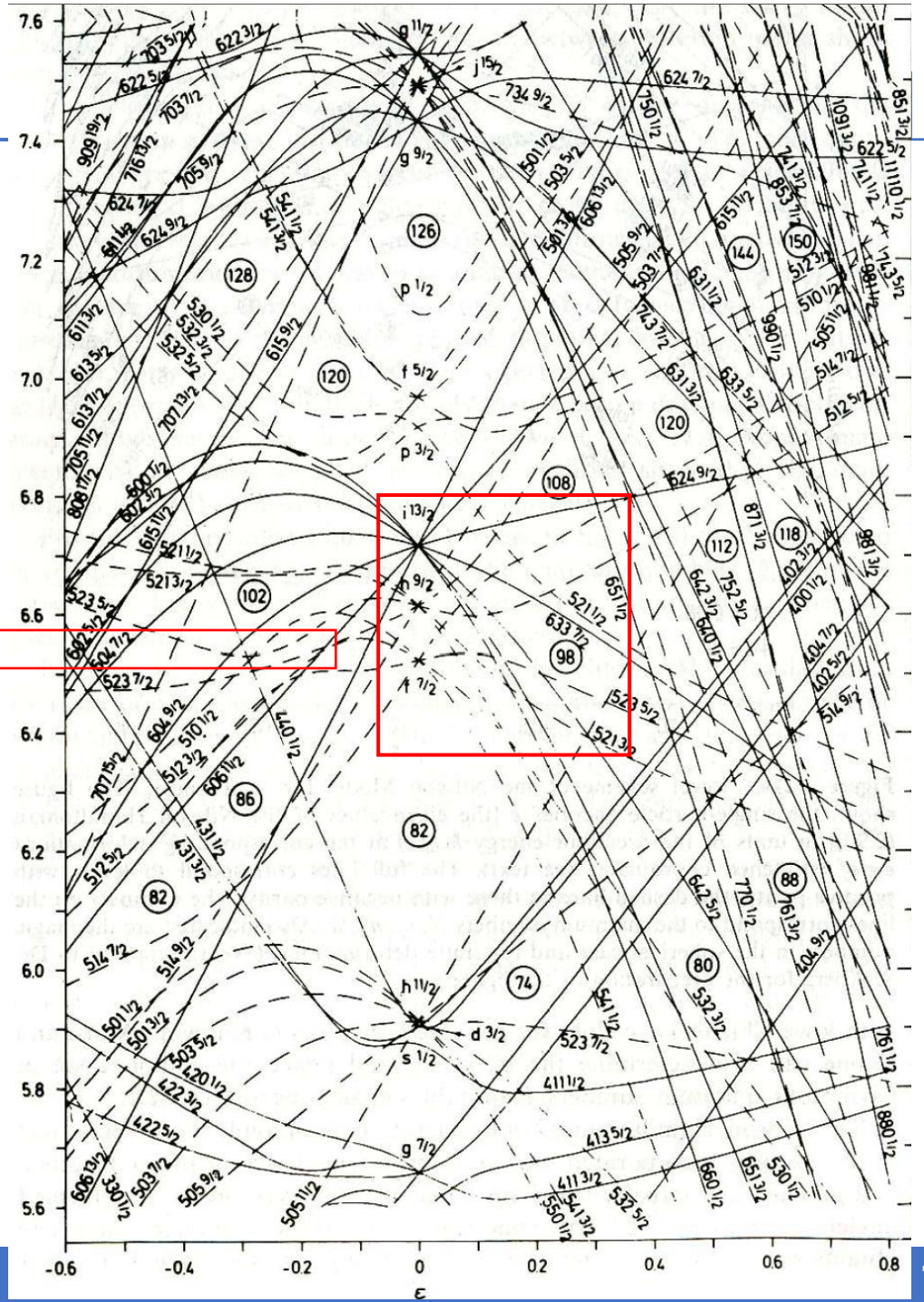
Neutron separation

Calculated by S. Dutta

Nilsson diagram for neutrons in rare earth



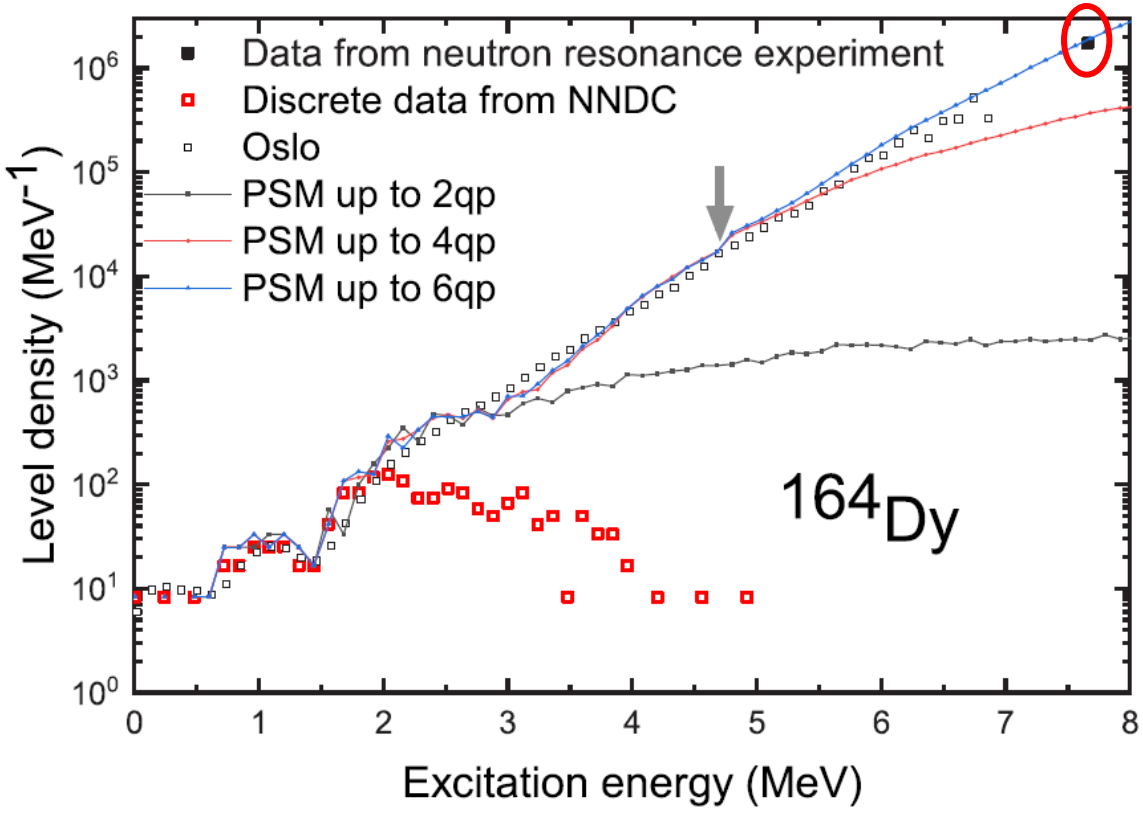
Deformed neutron Nilsson single-particle states



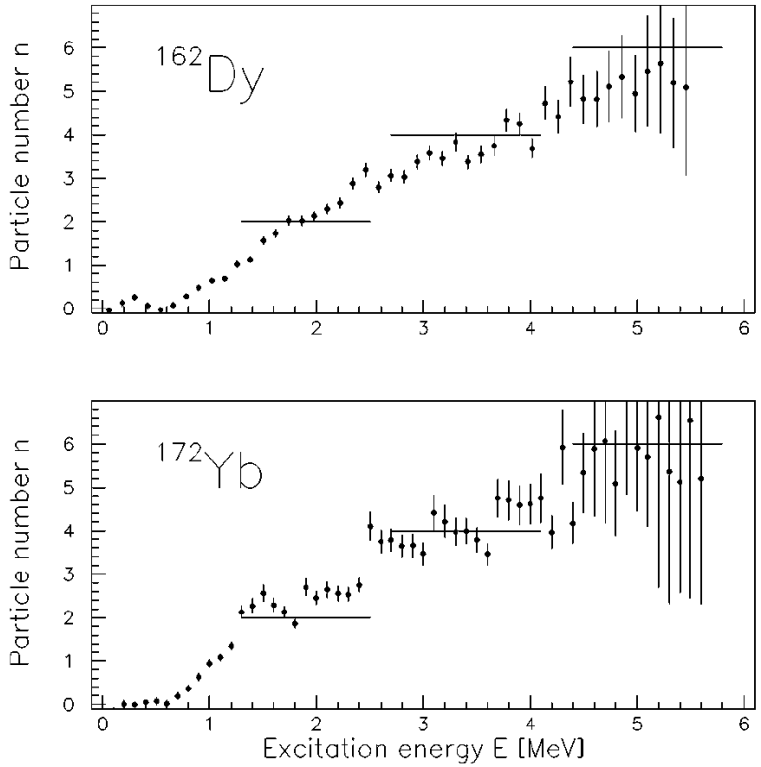
Level density in ^{164}Dy: testing our algorithm

- We reproduce both **low-energy spectra** and **high-energy data** from neutron resonance spacing measurement.
- One more qp configuration → one order of magnitude in NLD, which correlates well with the Oslo data.

J.-Q. Wang et al, Phys. Rev. C, 108, 034309 (2023)



M. Guttormsen, et al., Phys. Rev. C 62, 024306 (2000).

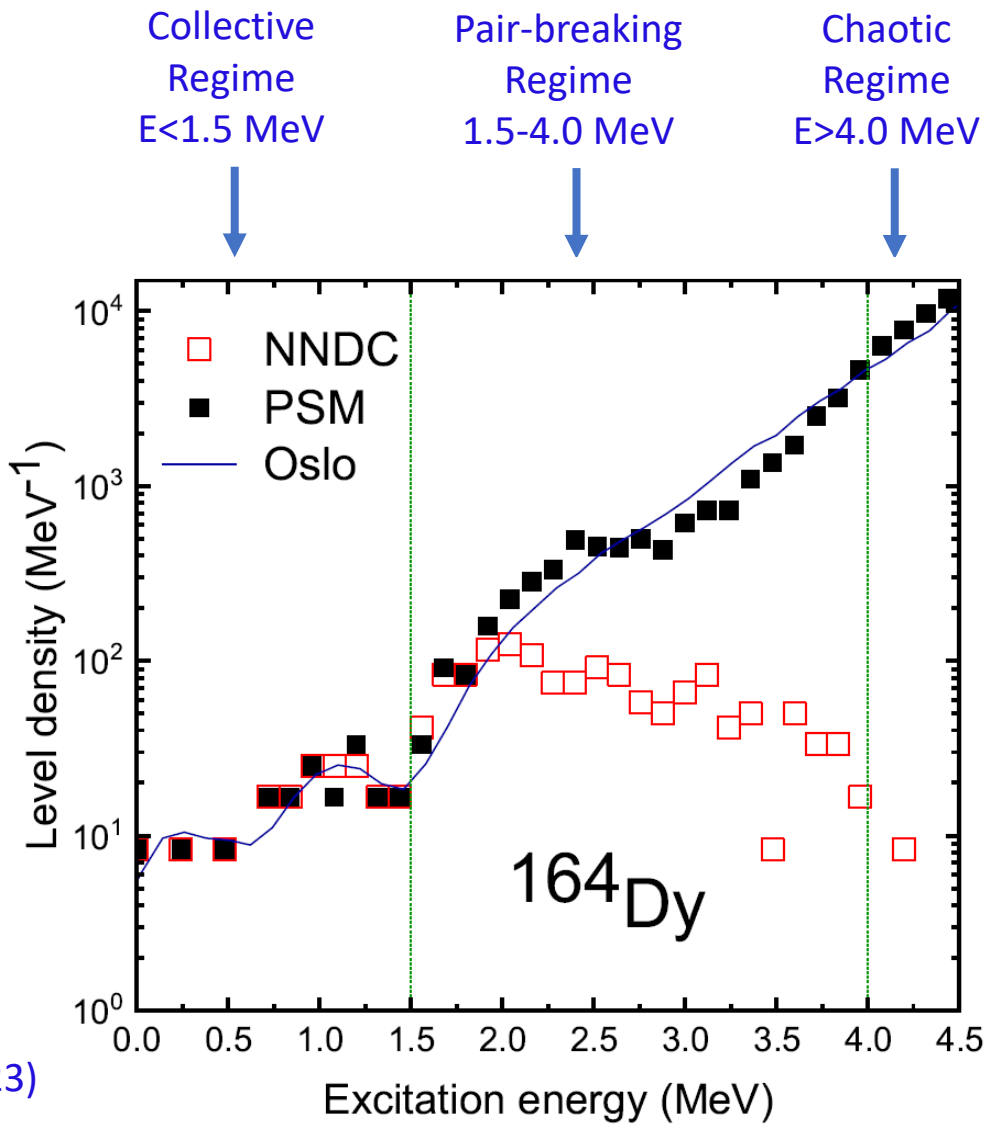


Nuclear level density: Structure effect

- The idea of using broken-pair configurations is based on the role of pairing correlation in nuclear structure and its decisive function in the formation and breaking of nuclear pairs

  V. Zelevinsky and M. Horoi, Prog. Part. Nucl. Phys. 105, 180 (2019).

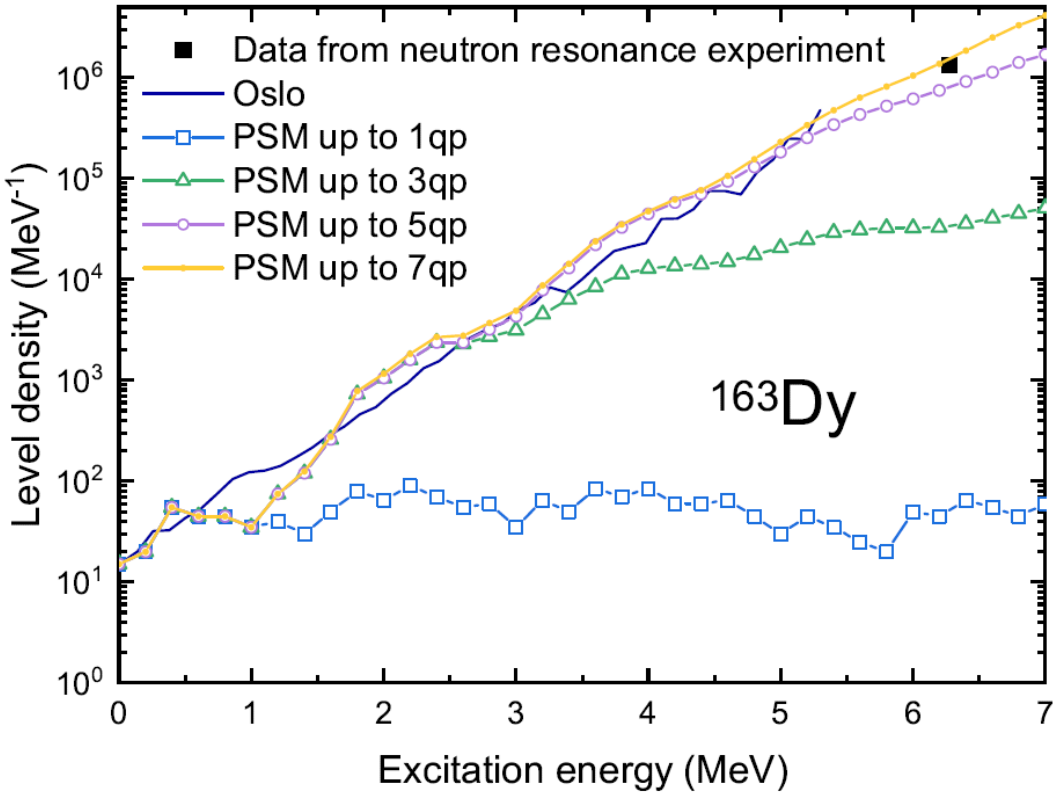
- The evolution of the excitation can be divided into three regimes:
- **Collective regime:** a few known collective bands.
- **Pair-breaking regime:** two rises – one pair breaking (2-qp states) and simultaneous breaking of a neutron- and a proton-pair (4-qp states).
- **Multi-qp regime:** more pairs are broken. Regular band structures may not be dominant.

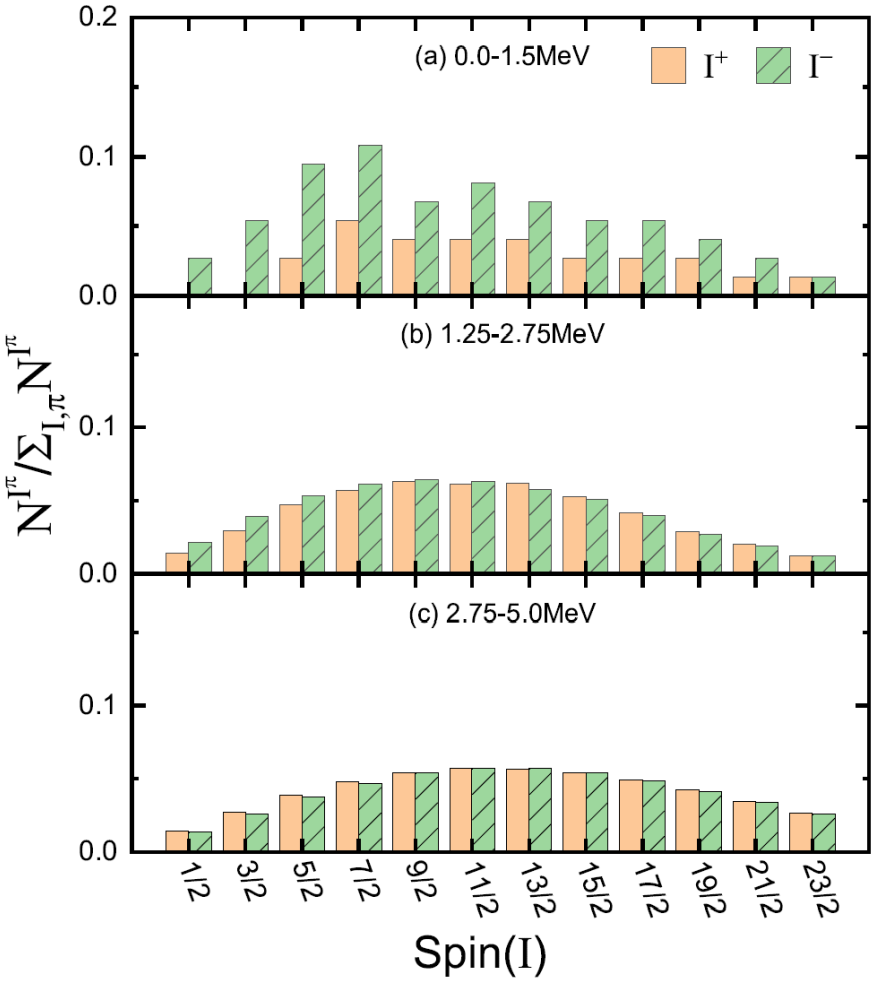


J.-Q. Wang et al,
Phys. Rev. C, 108, 034309 (2023)

Level density for odd-mass nuclei: ^{163}Dy

- Taking ^{163}Dy as example for odd-mass nuclei compared with Oslo NLD curve.
- It reproduces both low-energy spectra and high-energy data from neutron resonance spacing measurement.
- Spin distributions show a better Gaussian for E > 2.75 MeV.





J.-Q. Wang, et al., PRC 111, 034324 (2025)

Angular momentum problem in nuclear level density

- **Angular momentum** is one of the most frequently discussed quantities for nuclear states.
- In nuclear shell model, energy levels of definite quantum numbers, $E(I, \pi)$, can be calculated.
- In spectroscopic measurements, angular momentum of nuclear levels can usually be determined.
- When a nucleus is excited to a sufficiently high energy, average spacing between levels in medium to heavy nuclei can reach 10^{-3}–10^{-6} MeV, which are too dense to be resolved individually.
- Energy levels are described statistically by **level density.**
- In 1936, Hans Bethe laid the foundation for calculating nuclear level density (NLD) based on the **Fermi gas model**. Bethe's derivation gives **NLD with mixed spin *I*.**
- Based on Bethe's work, Ericson's later work provided an explicit expression for **spin distribution.**

$$\rho(E, I) \sim \rho(E) \frac{2I+1}{2\sigma^2} \exp\left[-\frac{I(I+1)}{2\sigma^2}\right]$$

H. A. Bethe, Phys. Rev. 50, 332 (1936).
T. Ericson, Advances in Physics, 9, 425 (1960).

Spin cutoff in NLD is still a problem

- The Fermi gas model contains no knowledge of nuclear structure.
- In Ericson's formula, **parameter σ** absorbs all nuclear structure effects. It determines how rapidly the exponential suppresses high-spin states. **σ** is commonly referred to as **spin cutoff** parameter.

$$\rho(E, I) \sim \rho(E) \frac{2I+1}{2\sigma^2} \exp\left[-\frac{I(I+1)}{2\sigma^2}\right]$$

- Ericson left **σ undefined.** Since then, **spin distribution** becomes a question **outside the model.**
- Landau first hypothesized that a complex nucleus is analogous to a rigid rotor with an effective moment of inertia, and total angular momentum of a nucleus is provided by collective rotation of **all nucleons** constituting the rotor.
- Extensive effort has been devoted to empirically fitting energy-dependent σ using experimental level schemes, but **we still do not know how the angular momentum are created.**

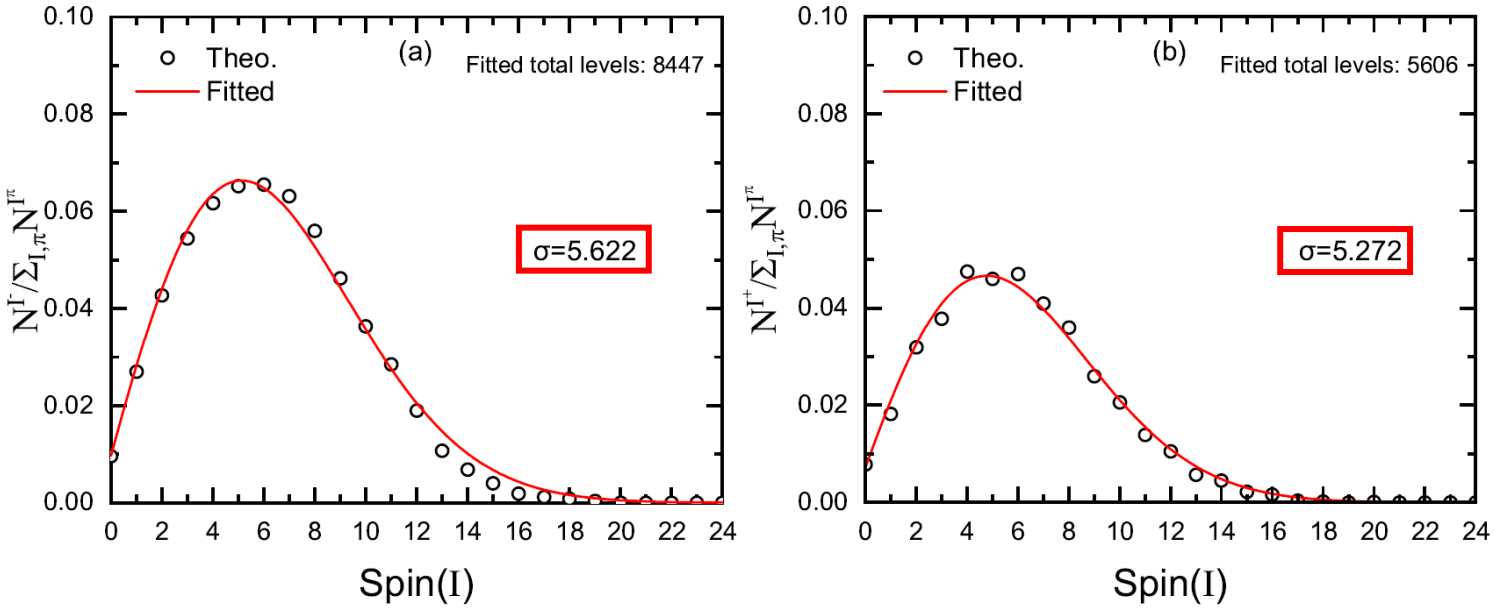
T. von Egidy and D. Bucurescu, Phys. Rev. C 80, 054310 (2009).
S. M. Grimes, T. N. Massey, and A. V. Voinov, Phys. Rev. C 99, 064331 (2019).

Energy-dependent spin distribution

- When using our levels to build level density, we have precise information on **spin and parity.**
- Taking the multi-qp regime (4.5~5.5 MeV) as example, our calculation indicates a Gaussian distribution as suggested by Ericson.
- The Ericson formula contains a **spin-cutoff parameter** σ, which can now be obtained by fitting our shell-model results to the Ericson formula.

$$\rho(E,I) \sim \rho(E)\ \frac{2I+1}{2\sigma^2}\exp\left[-\frac{I(I+1)}{2\sigma^2}\right]$$

T. Ericson, Advances in Physics 9, (1960) 425



Putting structure information into statistics: derivation of σ

- In another work, we ask: **Can we directly calculate spin cutoff parameter σ** in the statistical model?
- To do this, we must find out what was missing in Bethe's 1936 derivation.

- Bethe's NLD model and Ericson's spin distribution were based on the assumption that each nucleon is **independently and identically** distributed and the total angular momentum arises from **random coupling** of the angular momenta of individual particles.
- However, nuclei are **finite quantum systems** and coupling of angular momentum must obey strict coupling rules. **This is the place that needs to be fixed.**

- By definition, σ is standard deviation of the M distribution, and **σ^2 = Var(M)** is variance of M, where $M = m_1 + m_2 + \cdots m_N$ is the total z-component of angular momentum.
- When sampling variables, no repeated m values are allowed (required by Pauli principle). Statistically, this corresponds to **sampling without replacement.**

- Since $\mathrm{Var}(M) = \sum_i \mathrm{Var}(M_i)$ the term to be fixed is Var (M_i).

J.-C. Guo and Y. Sun, submitted for publication.

Angular momentum coupling based on projection operator

J.-C. Guo and Y. Sun, Phys. Rev. C 112, 064307 (2025).

- **Angular momentum projection (AMP)** is known to restore rotational symmetry in **deformed basis**.
- We find it can also **construct angular momentum** of many-nucleon states in multiple **spherical j-shells.**
- We have mathematically proven that construction of angular momentum can be done in a single spherical j-shell by calculating norm matrix $\langle\phi_i|\,\hat{P}^J_{MM}\,|\phi_j\rangle$, where $\{|\phi_i\rangle\}$ is J-mixed states and

$$\hat{P}^J_{MK} = \frac{2J+1}{8\pi^2}\int d\Omega\; D^J_{MK}(\alpha,\beta,\gamma)\hat{R}(\alpha,\beta,\gamma)$$

  is **J-projection operator.**

- The J-projection operator distinguishes the role of single-j and multi-j shells in angular momentum coupling. The new method **changes the traditional way of thinking** for construction of angular momentum of fermionic quantum many-body states.
- This new method may be applicable in any many-nucleon problems (**including reaction theory** and **fission theory**) where (1) the problem requires good total angular momentum, and (2) the spherical model basis involves multiple j shells.

Putting structure information into statistics: derivation of σ

- Our model space consists of multiple spherical j-shells, each characterized by single-particle energy ε_i, angular momentum j_i, with degeneracy $d_i = 2j_i + 1$.
- To correct Var(M_i), the statistical treatment is to introduce a so-called **finite population correction (FPC).**

$$\mathrm{Var}(M_i) = n_i \mathrm{Var}(m) \times \boxed{\frac{d_i - n_i}{d_i - 1}} \longleftarrow \textbf{FPC}$$

$$= n_i \mathrm{Var}(m) \times \frac{2j_i + 1 - n_i}{2j_i}$$

$$= \frac{1}{6} n_i (j_i + 1)(2j_i + 1 - n_i)$$

- FPC depends on degeneracy d_i and the average particle number $n_i = \dfrac{d_i}{e^{\beta(\varepsilon_i - \mu)} + 1}$ put n_i in the above equation, we obtain

$$\mathrm{Var}(M) = \sum_i w_i v_i \qquad \text{with } w_i = \frac{e^{\beta(\varepsilon_i - \mu)}}{\left[e^{\beta(\varepsilon_i - \mu)} + 1\right]^2} \text{ and } v_i = \frac{1}{6}(j_i + 1)(2j_i + 1)^2$$

J.-C. Guo and Y. Sun, submitted for publication.

Putting structure information into statistics: derivation of σ

- After some derivations, we can write $w(\varepsilon) = -k_B T \frac{df(\varepsilon)}{d\varepsilon}$, where $f(\varepsilon_i) = \frac{1}{e^{\beta(\varepsilon_i - \mu)} + 1}$ is the standard Fermi-Dirac distribution.

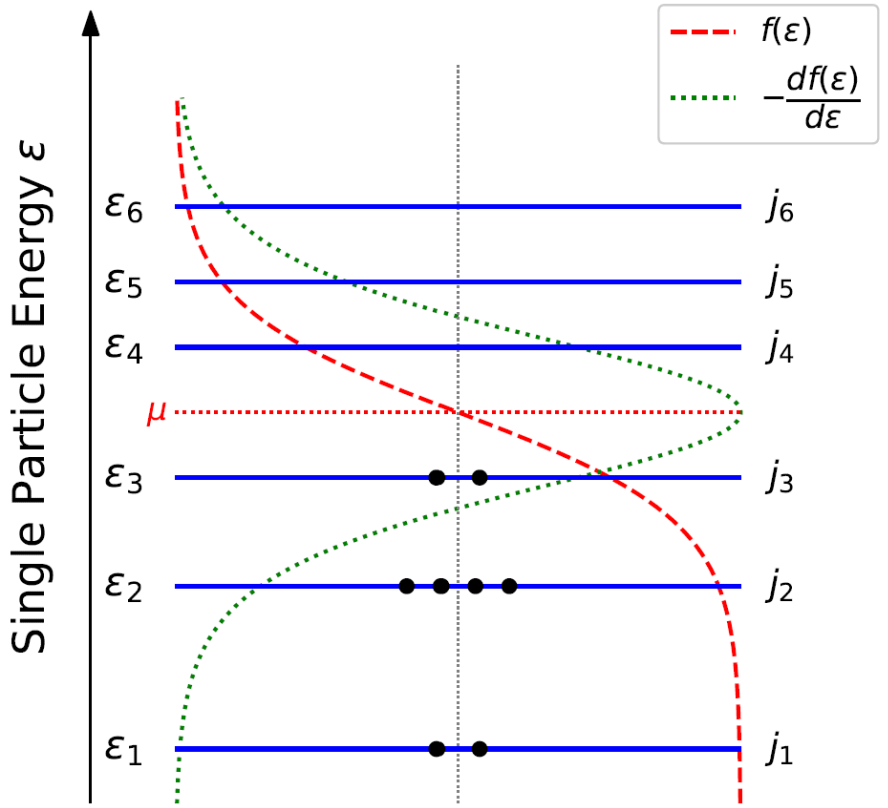
- The figure schematically plots a set of single-particle levels ε_j , along with f(ε), its derivative −df(ε)/dε , and chemical potential μ.

$$\mathrm{Var}(M) = \sum_i w_i v_i$$

- Among all j_i-shells in the model space, only those having overlap with the **bell curve** contribute to above summation.

$$v_i = \frac{1}{6}(j_i + 1)(2j_i + 1)^2$$

- In addition, if the contributed shells have a large j, $\sigma^2 = \mathrm{Var}(M)$ have a large value.
- We thus obtain an analytical solution for **$\sigma^2 = \mathrm{Var}(M)$.**



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Calculated examples for σ

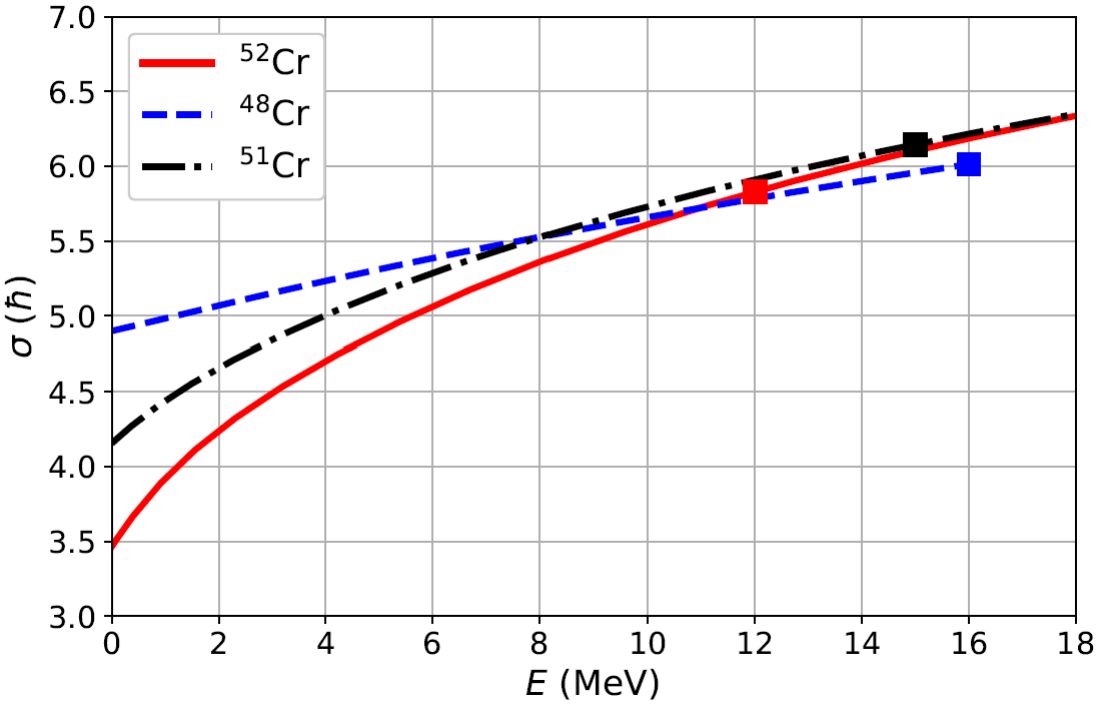


TABLE I. Single particle energies taken from Ref. [5].

Shells	$2s_{1/2}$	$1d_{3/2}$	$1f_{7/2}$	$2p_{3/2}$	$2p_{1/2}$	$1g_{9/2}$	$1f_{5/2}$
Energy (MeV)	−7.0	−5.0	0.0	2.0	4.0	4.5	5.5

[5] K. Kaneko, et al., Phys. Rev. C 78, 064312 (2008).

These examples include a well-deformed self-conjugate nucleus (^{48}Cr) as well as a doubly even (^{52}Cr) and a odd-neutron (^{51}Cr) semi-magic ones. **The single particle energies in Table I are the sole input to the calculation.**

- **Final remarks:** As this derivation is still based on sampling variables, we do not expect it work well for low energy regions (say, E ≤ 2 MeV) because when discrete energy levels dominate the structure, statistical descriptions generally break down.

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Summary

- Traditionally, low-energy discrete levels and chaotic regions are described separately. Knowledge on the door-way states has been missing.
- We tried to approach the door-way from both directions: Extending shell-model calculations to highly excited states and putting structure information into statistical models.
- We attempt to calculate a “complete” set of nuclear levels, with well-defined spin-parity, for arbitrarily heavy, deformed nuclei, from shell-model diagonalization calculations.
- Traditional statistical quantities, such as nuclear level density and γ-strength function, can be obtained up to and beyond the neutron separation energy by using these shell-model states.
- In a separate attempt to derive the spin cutoff parameter in Ericson formula, we find a missing term in statistical definition of σ, finite population correction.
- Analyzing the derived spin cutoff parameter, we conclude that the angular momentum in level density is primarily generated by states near the Fermi surface, while contributions from internal states can be neglected.

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