Low-Energy Limits: S factors, Scattering Lengths,...

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The Low-Energy Limit

- Why?
- The Penetration Factor
- Astrophysical S Factor
- Scattering Lengths and Effective Range

Why Consider the Low-Energy Limit?

- Astrophysics: $kT \approx$ a few keV in the cores of stars.
- Nuclear reactors: range of temperatures, but $kT = 0.0256$ eV @ 25°C.
- Simplicity (theorists like scattering lengths).

Penetration Factor

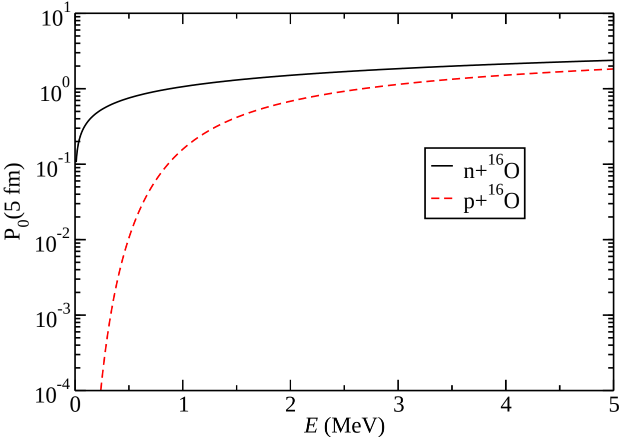
- Define $\rho = ka$, $k = \sqrt{2\mu E/\hbar^2}$, $\eta k = \alpha = Z_1 Z_2 e^2 \mu/\hbar^2$, $x = (8\eta\rho)^{1/2} = (8\alpha r)^{1/2}$ – which is E independent.
- And $2\pi\eta = \sqrt{E_G/E}$, where $E_G = 2(\pi Z_1 Z_2 e^2/\hbar)^2\mu$.
- Recall $P_\ell = \rho/[F_\ell^2(\rho) + G_\ell^2(\rho)]$.
- Low-Energy limits:

$$P_\ell \quad \to \quad \frac{\rho^{2\ell+1}}{(2\ell-1)!!} \quad \propto \quad E^{\ell+1/2} \qquad\qquad Z_1 Z_2 = 0$$

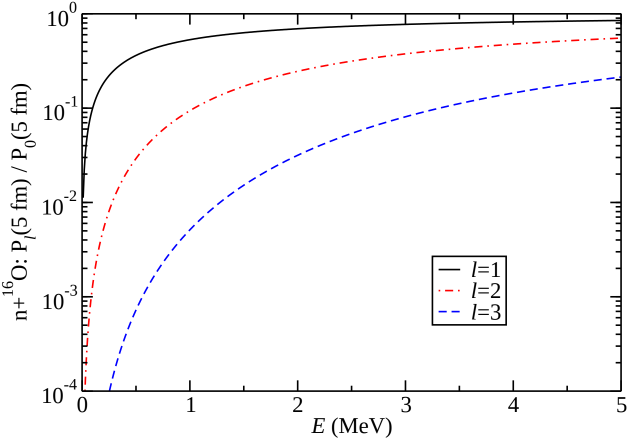
$$P_\ell \quad \to \quad \frac{\pi \exp(-2\pi\eta)}{4K_{2\ell+1}^2(x)} \qquad\qquad Z_1 Z_2 \neq 0.$$

- Presence of Coulomb *qualitatively* changes the limit!
- Lane and Thomas: “Well-known paradox.”

P_0 for nucleon + ^{16}O @ $a = 5$ fm

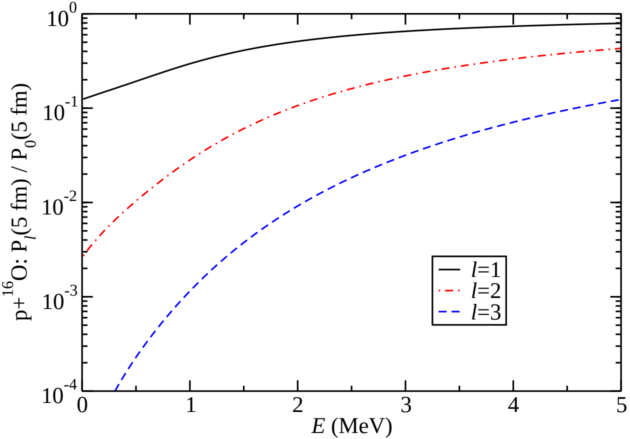


Angular Momentum Barrier for $n + {}^{16}\mathrm{O}$



Note that ratio $\to 0$ for $E \to 0$.

Angular Momentum Barrier for $p + {}^{16}\mathrm{O}$



Note that ratio is finite for $E \to 0$.

Low-Energy Limit of Reaction Cross Sections

- Recall

$$\sigma_{\alpha\alpha'} = \frac{\pi}{k_\alpha^2} \sum_{J,c\ni\alpha,c'\ni\alpha'} \frac{2J+1}{(2J_{\alpha 1}+1)(2J_{\alpha 2}+1)} |S^J_{c'c}|^2$$
$$= \sum_{J,c\ni\alpha,c'\ni\alpha'} \sigma^J_{cc'} \quad \text{and}$$
$$\boldsymbol{S} = \boldsymbol{\Omega}\left[\boldsymbol{1} + 2i\boldsymbol{P}^{1/2}[\boldsymbol{1} - \boldsymbol{R}(\boldsymbol{L}-\boldsymbol{B})]^{-1}\boldsymbol{R}\boldsymbol{P}^{1/2}\right]\boldsymbol{\Omega}.$$

- Assuming α' is open and neglecting the energy dependences of $[\boldsymbol{1} - \boldsymbol{R}(\boldsymbol{L}-\boldsymbol{B})]^{-1}\boldsymbol{R}$ and $P_{c'}$, we have

$$\sigma^J_{cc'} \propto \frac{P_c}{E_\alpha}.$$

Low-Energy Limit of Neutron-Induced Reaction Cross Sections

- Generally speaking, only $\ell_c = 0$ will contribute. Then

$$\sigma^J_{cc'} \propto \frac{P_c}{E_\alpha} \propto E_\alpha^{-1/2}. \tag{1}$$

- This is known as the "$1/v$" Law.
- It implies the cross section may be very large at low energies.
- The cross sections for ^{3}He$(n,p)^3$H, ^{6}Li$(n,t)\alpha$, and ^{10}B$(n,\alpha)^7$Li are 1000s of barns at $E = 0.0256$ eV.

Low-Energy Limit of Charged-Particle Reaction Cross Sections

- Ignoring the energy dependences due to resonance effects and $P_{c'}$, we have

$$\sigma^J_{cc'} \propto \frac{P_c}{E_\alpha} \propto \frac{\exp(-2\pi\eta_\alpha)}{E_\alpha} \quad \text{for all } \ell_c. \tag{2}$$

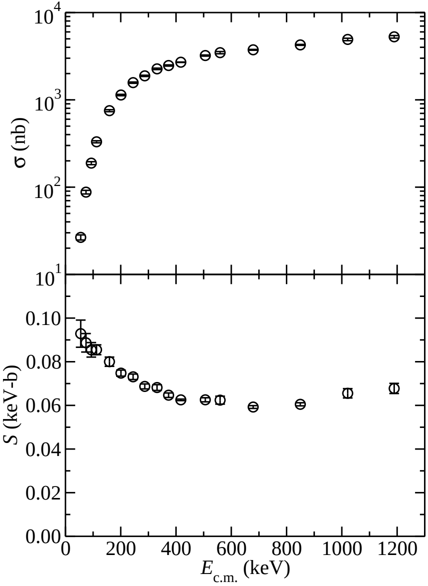
- Define the Astrophysical S Factor $S(E)$:

$$\sigma(E) = \frac{S(E)}{E}\exp(-2\pi\eta),$$

  the particle-pair label α has been dropped.

- $S(E)$ contains normalization and residual nuclear effects.
- $S(E) \to$ constant, as $E \to 0$.

Example of S Factor



$^3H(\alpha, \gamma)^7Li$ data from CRB's Ph.D. thesis.

Low-Energy Limit of the Neutron Elastic Scattering Cross Sections

- In this case, we have

$$\sigma_{\alpha\alpha} = \frac{\pi}{k_\alpha^2} \sum_{J,c\ni\alpha,c'\ni\alpha} \frac{2J+1}{(2J_{\alpha 1}+1)(2J_{\alpha 2}+1)} |\delta_{c'c} - S^J_{c'c}|^2.$$

- Limiting ourselves to ℓ_c, $\ell_{c'} = 0$, we find that

$$\sigma_{\alpha\alpha} \propto \text{constant}$$

  as $E \to 0$.

- This has important consequences for applications neutron transport (diffusion) in applications.

Other Limiting Cases

- The low-energy limit of charged-particle elastic scattering is the Rutherford Formula.
- A similar analysis can be performed for the case of crossing the threshold of a reaction channel.
- For example, when the threshold for (X, n) is crossed, one expects

$$\sigma_{Xn} \propto (E - E_0)^{1/2} \qquad \text{for } E \geq E_0,$$

  where E_0 is the threshold energy.

Effective Range Theory

- Another approach to short-range potentials is Effective Range Theory, developed by Bethe, Schwinger, Blatt, Jackson... See H.A. Bethe, Phys. Rev. **76**, 38 (1949).
- For $\ell = 0$ and no Coulomb, this results in

$$k \cot \delta_c = -\frac{1}{a_e} + \frac{1}{2} r_e k^2 + \ldots \equiv K(E).$$

- For $\ell = 0$ and Coulomb, this becomes

$$C_0^2 k \cot \delta_c + 2\eta k h(\eta) = -\frac{1}{a_e} + \frac{1}{2} r_e k^2 + \ldots \equiv K(E),$$

where $h(\eta) = \frac{1}{2}[\Psi(1 + i\eta) + \Psi(1 - i\eta)] - \log \eta$ and Ψ is the digamma function.

Effective Range Theory, Continued

- The quantity a_e is known as the scattering length and r_e as the effective range.
- $K(E)$ is analytic near $E = 0$, and can be modeled using a Taylor series or Padé approximant.
- Effective Range Theory can be extended to $\ell > 0$ [H. van Haeringen, J. Math. Phys. **18**, 927 (1977)] and multi-channel situations [M.H. Ross and G.L. Shaw, Annals of Physics, **13**, 147 (1961)], but applications are limited.

Scattering Lengths

- The zero-energy limit of the neutron elastic scattering cross section is given by

$$\sigma = 4\pi \sum_s \frac{2s+1}{(2J_{\alpha 1}+1)(2J_{\alpha 2}+1)} [a_e(s)]^2,$$

  where s is channel spin.

- Scattering lengths for neutrons can be considered an observable quantity (effectively).
- For charged-particle elastic scattering, the nuclear phase shifts are overwhelmed by the Rutherford amplitude. In this case, scattering lengths can only determined by extrapolation from higher energies, where the nuclear effects are measureable. They are thus not observable in the same sense.

Scattering Lengths and the Asymptotic Wavefunction

- For $E > 0$ and large r, we have

$$u(r) \propto \cos\delta_c F_\ell(kr) + \sin\delta_c G_\ell(kr).$$

- For $E = 0$, $\ell = 0$, no Coulomb, and large r, we have

$$u(r) \propto 1 - \frac{r}{a_e}.$$

- For $E = 0$, $\ell = 0$, Coulomb, and large r, we have

$$u(r) \propto x[I_1(x) - 4\alpha a_e K_1(x)],$$

where $\alpha = Z_1 Z_2 e^2 \mu/\hbar^2$ and $x = (8\eta\rho)^{1/2} = (8\alpha r)^{1/2}$.

Scattering Lengths and R Matrix

- For a single-channel R matrix

$$\exp(2i\delta_c) = \exp(-2i\phi_c)[1 + 2iP_c\boldsymbol{\gamma}_c^T\boldsymbol{A}\boldsymbol{\gamma}_c].$$

- For neutrons, $E \to 0$ yields

$$a_e = a(1 - \boldsymbol{\gamma}_c^T\boldsymbol{A}\boldsymbol{\gamma}_c).$$

- For charged particles, we have

$$a_e = a\left[\frac{2I_1(x)}{x^2K_1(x)} - \frac{\boldsymbol{\gamma}_c^T\boldsymbol{A}\boldsymbol{\gamma}_c}{x^2K_1^2(x)}\right].$$

- Here, a is the channel radius. We must keep our a's straight here!

Summary Comments on Effective Range Theory

- Effective Range Theory is most commonly encountered in (very) few-body physics.
- Neutron scattering lengths are also very important in applications of nuclear physics.
- Both Effective Range Theory and the phenomenological R-matrix can be used to fit scattering data. In general, one expects that a similar number of free parameters would provide a similar quality of fit.
- Effective Range Theory is not the *natural* choice for describing resonances or multi-channel phenomena.
- The two approaches are easily related at $E = 0$.

Thank you for your attention.