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INSTITUTE OF NUCLEAR & PARTICLE PHYSICS

Single Channel Scattering with charged particles in momentum space

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Lecture 5

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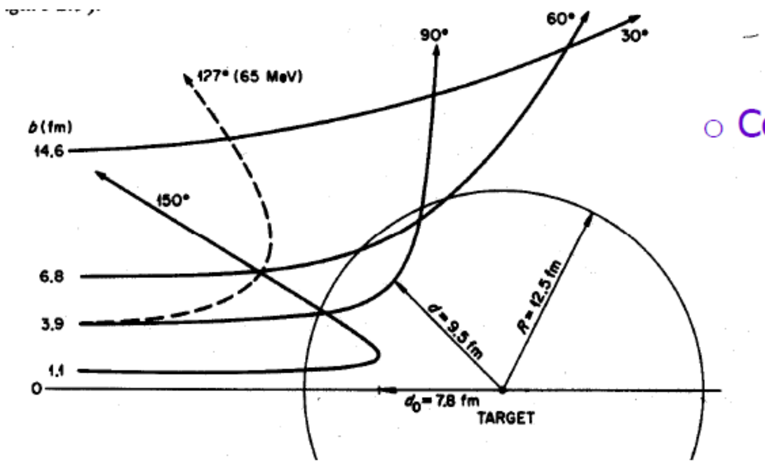
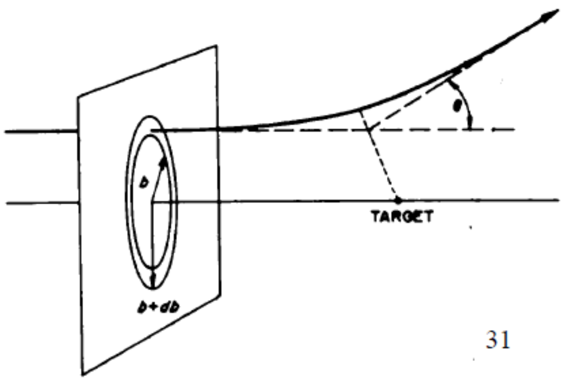
NERSC
OHIO UNIVERSITY

Classical Coulomb Scattering



- Coulomb trajectories are hyperbolas

$$\tan\frac{\theta}{2} = \frac{\eta}{bk}$$



Rutherford formula for Coulomb cross section

$$\sigma(\theta) \equiv \frac{b(\theta)}{\sin\theta}\frac{\mathrm{d}b}{\mathrm{d}\theta} = \frac{\eta^2}{4k^2\sin^4(\theta/2)}$$

Preliminaries: Gellman-Goldberger Relation or two-potential formula

Interaction between projectile and target decomposes into two parts:

$$V = V_0 + V_1$$

Division is particularly useful if the scattering wave function with respect to V_0 can be treated exactly.

$$\begin{aligned} T &= (V_0 + V_1) + (V_0 + V_1)\, G_0 T \\ &= (V_0 + V_1) + V_0\, G_0 T + V_1\, G_0 T \end{aligned}$$

Reminder:

$$|\,\vec{p}\,\rangle^{(+)} = |\,\vec{p}\,\rangle + G_0 V\; |\,\vec{p}\,\rangle^{(+)}$$

or:

$$|\,\vec{p}\,\rangle^{(+)} = |\,\vec{p}\,\rangle + G_0 V\; |\,\vec{p}\,\rangle^{(+)}$$

From Low equation:

$$|\,\vec{p}\,\rangle^{(+)} = |\,\vec{p}\,\rangle + GV\; |\,\vec{p}\,\rangle = (1 + GV)\; |\,\vec{p}\,\rangle = \Omega^{(+)}\; |\,p\rangle$$

$\Rightarrow$ different representations of the Møller operator

$$\Omega^{(+)} = (1 - G_0 V)^{-1} = (1 + GV) = 1 + G_0 T$$

multiply

$$\begin{aligned} T &= (V_0 + V_1) + (V_0 + V_1)\, G_0 T \\ &= (V_0 + V_1) + V_0\, G_0 T + V_1\, G_0 T \end{aligned}$$

from the left with $\Omega_0^{(+)\dagger} := (1 - V_0 G_0)^{-1}$

$$\begin{aligned} (1 - V_0 G_0)^{-1}\,(1 - V_0 G_0) T &= (1 - V_0 G_0)^{-1}\, V_0 + (1 - V_0 G_0)^{-1}\, V_1 \\ &+ (1 - V_0 G_0)^{-1}\, V_1 G_0 T \\ T &= T_0 + (1 - V_0 G_0)^{-1}\, V_1\, (1 + G_0 T) \end{aligned}$$

Where $T_0 = V_0 + V_0\, G_0 T_0$ exact solution of the Hamiltonian $H_0 + V_0$

$\Rightarrow$ $$T = T_0 + \Omega_0^{(-)\dagger}\, V_1\, \Omega^{(+)}$$ **exact**

Often used approximation: $T \approx T_0 + \Omega_0^{(-)\dagger}\, V_1\, \Omega_0^{(+)}$

distorted wave Born approximation.

How can we use the Gellman-Goldberger relation to compute with the Coulomb force in momentum space?

Define: $W = V^C + V^S$ V^S is an arbitrary short-ranged potential.

V^C is a **repulsive** Coulomb potential

The familiar two-potential formula for the scattering amplitude is then given as

$$\bar{f}(E,\theta) = \frac{-\eta(E)}{2k_0 \sin^2\frac{\theta}{2}} e^{-2i\eta(E)\ln(\sin\theta/2)} - \frac{2\pi^2\mu}{k_0}\sum_l (2l+1) e^{2i[\sigma_l(E)-\sigma_0(E)]} \langle k_0|\tau_l(E)|k_0\rangle P_l(\cos\theta)$$

$$\sigma_l(E) = \arg\Gamma(l+1+i\eta) \qquad \eta(E) = Z_1 Z_2 \mu/\hbar k_0.$$

$$\langle k_0|\tau_l(E)|k_0\rangle = (-2\pi^2\mu)^{-1} e^{i\delta_l(E)} \sin\delta_l(E)$$

Related to the Coulomb modified nuclear phase shift and solution of

$$\langle k'|\tau_l(E)|k\rangle = \langle k'|U_l|k\rangle + \int \langle k'|U_l|k''\rangle \frac{4\pi k''^2\, dk''}{E - E'' + i\varepsilon} \langle k''|\tau_l(E)|k\rangle$$

$$\langle k'|U_l|k\rangle = \langle (\phi_l^C)^{(+)}(k')|V^S|(\phi_l^C)^{(+)}(k)\rangle$$

To compute:

$$\langle k'|U_l|k\rangle = \langle (\phi_l^{\mathrm{C}})^{(+)}(k')|V^{\mathrm{S}}|(\phi_l^{\mathrm{C}})^{(+)}(k)\rangle$$

$$= (4\pi)^2 \mathcal{N}_l^*(k')\mathcal{N}_l(k) \int \underline{\langle \phi_l^{\mathrm{C}}(k')|k'''\rangle} k'''^2\, \mathrm{d}k''' \langle k'''|V^{\mathrm{S}}|k''\rangle k''^2\, \mathrm{d}k'' \langle k''|\phi_l^{\mathrm{C}}(k)\rangle$$

Coulomb functions in momentum space

or

$$\langle k'|U_l|k\rangle = \int \underline{\langle \phi_l^{\mathrm{C}}(k')|r'\rangle} r'^2\, \mathrm{d}r' \langle r'|V_l^{\mathrm{S}}|r''\rangle r''^2\, \mathrm{d}r'' \langle r''|\phi_l^{\mathrm{C}}(k)\rangle$$

Coulomb functions in coordinate space

A practical calculational method for treating Coulomb scattering in momentum space

Ch Elster†, L C Liu‡ and R M Thaler§

Chooses the latter

J. Phys. G: Nucl. Part. Phys. **19** (1993) 2123–2134.

Example: with Yukawa potential :

$$V^{S}(r) = U_0 \exp(-\lambda r) \text{ with } U_0 = -300 \text{ MeV}$$

$Z = 20$

T_p (MeV)	$V^C = 0$ r-space	$V^C = 0$ k-space	$V^C \neq 0$ r-space	$V^C \neq 0$ k-space
25	34.31	34.32	6.49	6.50
100	29.49	29.50	23.32	23.31
200	24.28	24.30	22.96	22.97
500	16.89	16.89	17.07	17.07
800	13.35	13.35	13.59	13.59

$\lambda = 770 \text{ MeV}\, c^{-1}$

$Z = 82$

T_p (MeV)	l	$V^C = 0$ r-space	$V^C = 0$ k-space	$V^C \neq 0$ r-space	$V^C \neq 0$ k-space
50	0	126.8	126.8	48.49	48.29
	1	97.84	97.84	22.70	22.62
	2	55.25	55.22	9.93	9.90
	3	23.75	23.73	4.39	4.38
	4	9.87	9.86	1.95	1.94
200	0	77.22	77.10	81.16	81.02
	1	66.25	66.14	62.15	62.06
	2	52.31	52.23	44.88	44.82
	3	38.79	38.73	31.26	31.22
	4	27.56	27.52	21.33	21.31
	5	19.09	19.06	14.41	14.39

$\lambda = 280 \text{ MeV}\, c^{-1}$

Challenge I: momentum space Coulomb functions

General: $$\psi^{C(+)}_{\mathbf{q},\eta}(\mathbf{p}) = \lim_{\gamma\to+0}\int d^3\mathbf{r}\, e^{-i\mathbf{p}\mathbf{r}-\gamma r}\,\psi^{C(+)}_{\mathbf{q},\eta}(r)$$

FT: A. Chan, MS thesis U. Waterloo (2007)

$$= -4\pi e^{-\pi\eta/2}\Gamma(1+i\eta)\lim_{\gamma\to+0}\frac{d}{d\gamma}\left\{\frac{[p^2-(q+i\gamma)^2]^{i\eta}}{[|\mathbf{p}-\mathbf{q}|^2+\gamma^2]^{1+i\eta}}\right\}$$

Partial wave decomposition (Mukhamedzanov, Dolinskii) (1966)

$$\psi^{C(+)}_{\mathbf{q},\eta}(\mathbf{p}) \equiv \sum_{l=0}^{\infty}(2l+1)\psi^{C}_{l,q,\eta}(p)P_l(\hat{\mathbf{p}}\cdot\hat{\mathbf{q}}), \qquad \psi^{C}_{l,q,\eta}(p) = \frac{1}{2}\int_{-1}^{1}dz\,P_l(z)\psi^{C(+)}_{\mathbf{q},\eta}(\mathbf{p}),$$

$$\frac{1}{2}\int_{-1}^{1}dz\,P_l(z)(\zeta-z)^{-1-i\eta} = \frac{e^{\pi\eta}}{\Gamma(1+i\eta)}(\zeta^2-1)^{-i\eta/2}Q_l^{i\eta}(\zeta)$$

$$\zeta \equiv \frac{p^2+q^2+\gamma^2}{2pq},$$

Essential:

$Q_l^{i\eta}(\zeta)$ has different representations depending on ζ

$Q_l^{i\eta}(\zeta)$ has different representations in terms of the hypergeometric function $_2F_1$ (a;b;c;z) depending on ζ

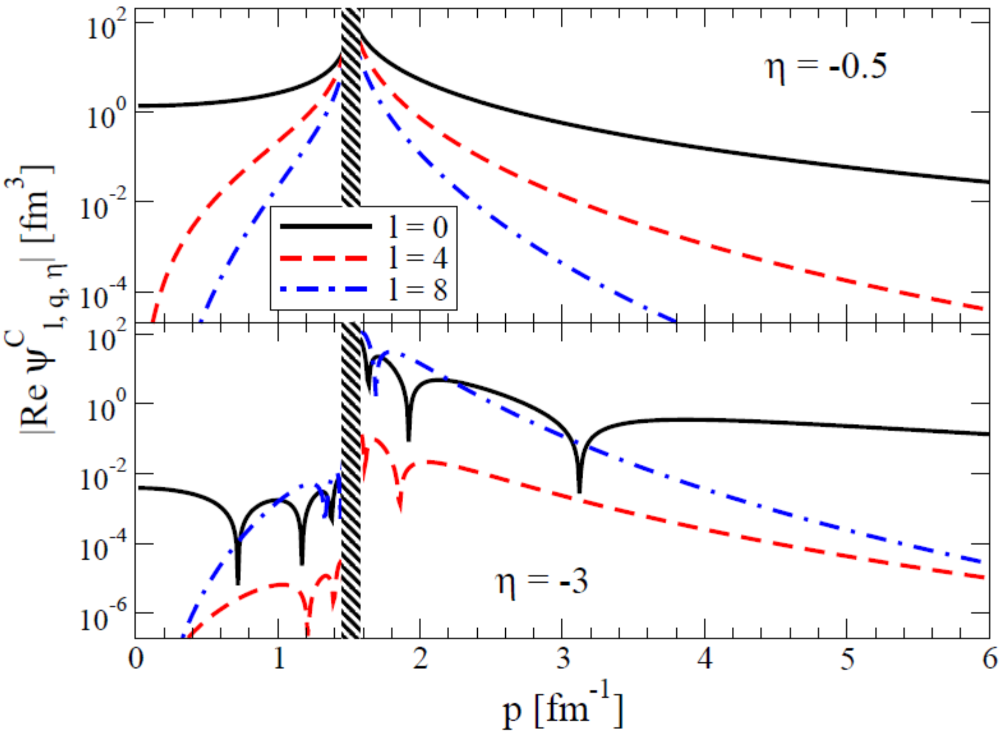
ζ large enough (p and q different) → **"regular" representation**

$$Q_l^{i\eta}(\zeta) = \frac{e^{-\pi\eta}\Gamma(l+i\eta+1)\Gamma(1/2)}{2^{l+1}\Gamma(l+3/2)}(\zeta^2-1)^{i\eta/2}\,\zeta^{-l-i\eta-1} \times {}_2F_1\left(\frac{l+i\eta+2}{2},\frac{l+i\eta+1}{2};l+\frac{3}{2};\frac{1}{\zeta^2}\right)$$

ζ ≈ 1 (p ≈ q) → **"pole-proximity" representation**

$$Q_l^{i\eta}(\zeta) = \frac{1}{2}e^{-\pi\eta}\left\{\Gamma(i\eta)\left(\frac{\zeta+1}{\zeta-1}\right)^{i\eta/2}{}_2F_1\left(-l,l+1;1-i\eta;\frac{1-\zeta}{2}\right) + \frac{\Gamma(-i\eta)\Gamma(l+i\eta+1)}{\Gamma(l-i\eta+1)}\left(\frac{\zeta-1}{\zeta+1}\right)^{i\eta/2}{}_2F_1\left(-l,l+1;1+i\eta;\frac{1-\zeta}{2}\right)\right\}$$

q = 1.5 fm^{-1}



Coulomb wave functions in momentum space☆

V. Eremenko [a,b,g,*], N.J. Upadhyay [c,d,*], I.J. Thompson [e], Ch. Elster [a,b,*], F.M. Nunes [c,d], G. Arbanas [f], J.E. Escher [e], L. Hlophe [a,b], On behalf of the TORUS Collaboration

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Challenge II:
Matrix elements with Coulomb basis functions

Separable t-matrix derived from p+A optical potential (generalized EST scheme)

$$t_l(E) = \sum_{i,j} u|f_{l,k_{E_i}}\rangle \tau_{ij}(E) \langle f^*_{l,k_{E_j}}|u$$

Nuclear matrix elements $\langle p|t_l(E)|p'\rangle$

$$\langle p|u|f_{l,k_E}\rangle = t_l(p, k_E; E_{k_E}) \equiv u_l(p)$$
$$\langle f^*_{l,k_E}|u|p'\rangle = t_l(p', k_E; E_{k_E}) \equiv u_l(p')$$

Coulomb distorted nuclear matrix element

$$\langle \psi^C_{l,p}|u|f_{l,k_E}\rangle = \int_0^\infty \frac{dq\, q^2}{2\pi^2}\, u_l(q)\psi^C_{l,p}(q)^\star \equiv u^C_l(p)$$
$$\langle f^*_{l,k_E}|u|\psi^C_{l,p}\rangle = \int_0^\infty \frac{dq\, q^2}{2\pi^2}\, u_l(q)\; \psi^C_{l,p}(q) \equiv u^C_l(p)^\dagger$$

"oscillatory" singularity at $q = p$: $\lim_{\gamma\to+0} \frac{1}{(q-p+i\gamma)^{1+i\eta}}$

Gel'fand-Shilov Regularization:

Generalization of Principal value regularization

Idea: reduce value of integrand near singularity

$$\int_{-\Delta}^{\Delta} dx \frac{\varphi(x)}{x^{1+i\eta}} = \int_{-\Delta}^{\Delta} dx \frac{\varphi(x) - \varphi(0) - \varphi'(0)x}{x^{1+i\eta}}$$

simplified

$$- \frac{i\varphi(0)}{\eta}[\Delta^{-i\eta} - (\Delta)^{-i\eta}] + \ldots$$

- **Reduce integrand around pole by subtracting 2 terms of the Laurent series**

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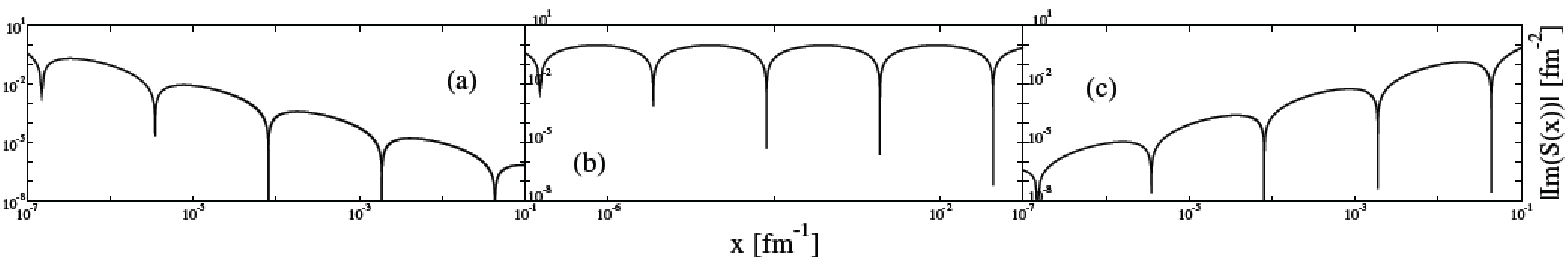
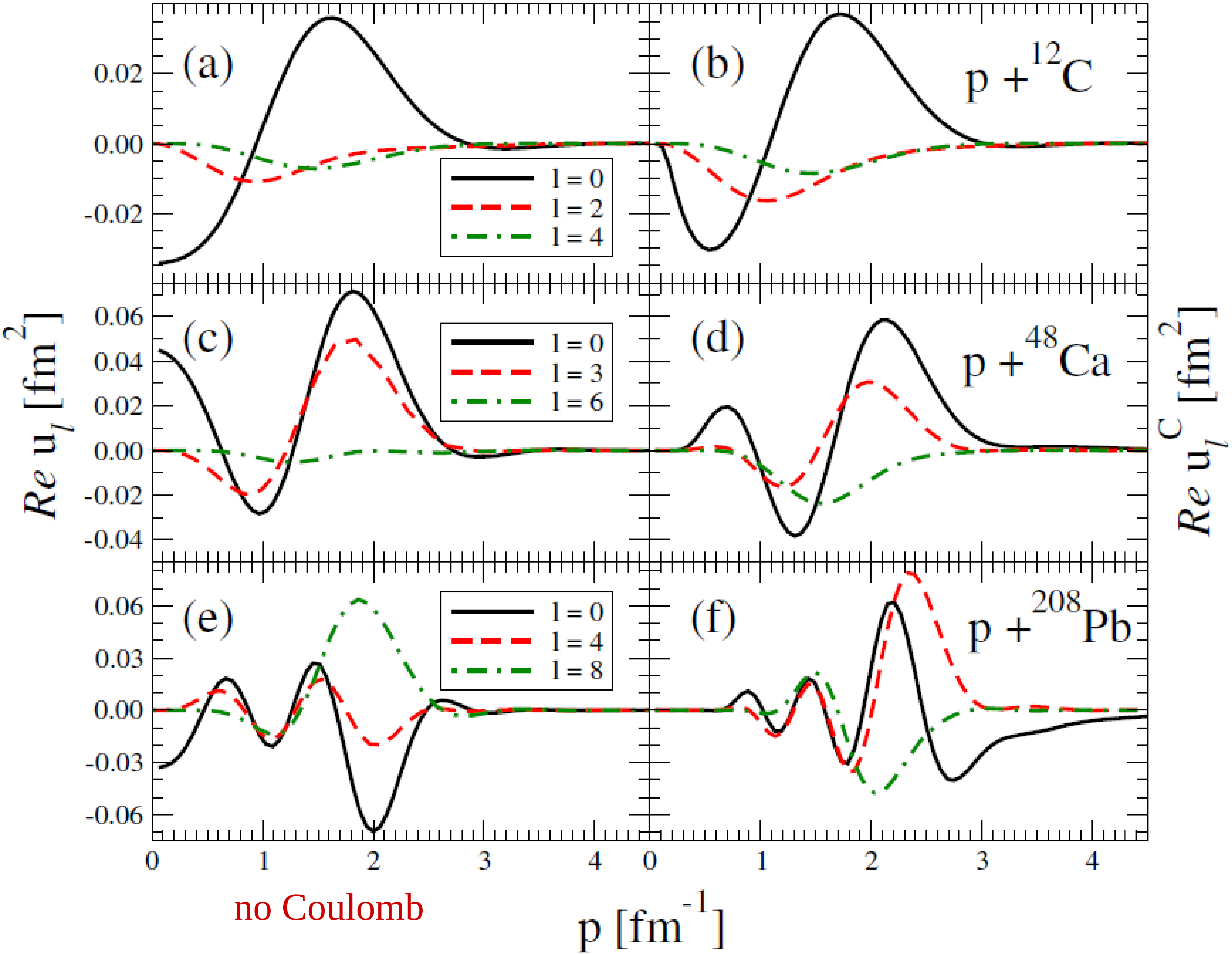


I. M. Gel'fand and G. E. Shilov. "Generalized Functions". Vol. 1. Academic Press, New York and London. 1964.


(a)
(b)
p + 12C
(c)
(d)
p + 48Ca
(e)
(f)
p + 208Pb
l = 0
l = 2
l = 4
l = 0
l = 3
l = 6
l = 0
l = 4
l = 8
Re u_l [fm^2]
Re u_l^C [fm^2]
p [fm^-1]
0.02
0.00
-0.02
0.06
0.04
0.02
0.00
-0.02
-0.04
0.06
0.03
0.00
-0.03
-0.06
0
1
2
3
4
0
1
2
3
4

no Coulomb