Coulomb functions

That $F_L(\eta,\rho)$ is regular means $F_L(\eta,\rho{=}0) = 0$, and irregularity means $G_L(\eta,\rho{=}0) \neq 0$. They are related by the Wronskian

$$G_L(\eta,\rho)\frac{\mathrm{d}F_L(\eta,\rho)}{\mathrm{d}\rho} - F_L(\eta,\rho)\frac{\mathrm{d}G_L(\eta,\rho)}{\mathrm{d}\rho} = 1$$

$$\text{or}\quad W(G,F) \equiv G\,F' - G'\,F = k. \tag{3.1.11}$$

Note that mathematics texts such as [1] usually define G' as $dG/d\rho$, but we denote this by $\dot{G}$. Since ρ is the dimensionless radius $\rho = kR$, we will use the prime for derivatives with respect to R, so $G' = k\dot{G}$, etc. The Wronskian is equivalently $G\dot{F} - \dot{G}F = 1$.

The Coulomb Hankel functions are combinations of F and G,

$$H_L^{\pm}(\eta,\rho) = G_L(\eta,\rho) \pm \imath F_L(\eta,\rho). \tag{3.1.12}$$

Coulomb functions for $\eta = 0$

The $\eta = 0$ functions are more directly known in terms of Bessel functions:

$$F_L(0,\rho) = \rho\, j_L(\rho) = (\pi\rho/2)^{1/2} J_{L+1/2}(\rho)$$

$$G_L(0,\rho) = -\rho\, y_L(\rho) = -(\pi\rho/2)^{1/2} Y_{L+1/2}(\rho), \tag{3.1.13}$$

where the irregular spherical Bessel function $y_L(\rho)$ is sometimes written as $n_L(\rho)$ (the Neumann function). The J_ν and Y_ν are the cylindrical Bessel functions. The $\eta = 0$ Coulomb functions for the first few L values are

$$F_0(0,\rho) = \sin\rho,$$

$$G_0(0,\rho) = \cos\rho; \tag{3.1.14}$$

$$F_1(0,\rho) = (\sin\rho - \rho\cos\rho)/\rho,$$

$$G_1(0,\rho) = (\cos\rho + \rho\sin\rho)/\rho; \tag{3.1.15}$$

$$F_2(0,\rho) = ((3-\rho^2)\sin\rho - 3\rho\cos\rho)/\rho^2,$$

$$G_2(0,\rho) = ((3-\rho^2)\cos\rho + 3\rho\sin\rho)/\rho^2. \tag{3.1.16}$$

Their behaviour near the origin, for $\rho \ll L$, is

$$F_L(0,\rho) \sim \frac{1}{(2L+1)(2L-1)\cdots 3.1}\rho^{L+1} \tag{3.1.17}$$

$$G_L(0,\rho) \sim (2L-1)\cdots 3.1\ \rho^{-L}, \tag{3.1.18}$$

Box 3.1 (*Continued*)
and their asymptotic behaviour when $\rho \gg L$ is

$$F_L(0,\rho) \sim \sin(\rho - L\pi/2)$$
$$G_L(0,\rho) \sim \cos(\rho - L\pi/2) \quad (3.1.19)$$
$$H_L^{\pm}(0,\rho) \sim e^{\pm\imath(\rho - L\pi/2)} = \imath^{\mp L} e^{\pm\imath\rho}. \quad (3.1.20)$$

So H_L^+ describes an outgoing wave $e^{\imath\rho}$, and H_L^- an incoming wave $e^{-\imath\rho}$.
Coulomb functions for $\eta \neq 0$ are described on page 62, and Whittaker functions on page 135.

Box 3.1 **Coulomb functions**

The spherical nature of the potential is crucial in allowing us to solve for each partial-wave function separately; this corresponds to angular momentum being conserved when potentials are spherical.

Equation (3.1.10) is a second-order equation, and so needs two boundary conditions specified in order to fix a solution. One boundary condition already known is that $\chi_L(0) = 0$. The other is fixed by the large R behavior, so that it reproduces the external form of Eq. (3.1.6). Since $f(\theta)$ is not yet known, the role of Eq. (3.1.6) is to fix the overall normalization of the $\chi_L(R)$. We show below how to accomplish these things.

As usual in quantum mechanical matching, both the functions and their derivatives must agree continuously. We therefore match interior and exterior functions and their derivatives at some *matching radius* $R = a$ chosen outside the finite range R_n of the nuclear potential.

Radial solutions for zero potential

For $R \geq a$ we have $V(R) = 0$, so at and outside the matching radius the radial wave functions must attain their external forms, which we name $\chi_L^{\text{ext}}(R)$. The free-field partial-wave equation may be simplified from Eq. (3.1.10), and rewritten using a change of variable from R to the dimensionless radius

$$\rho \equiv kR, \quad (3.1.21)$$

so the $\chi_L^{\text{ext}}(R)$ satisfy

$$\left[\frac{d^2}{d\rho^2} - \frac{L(L+1)}{\rho^2} + 1\right] \chi_L^{\text{ext}}(\rho/k) = 0. \quad (3.1.22)$$

[1] It implies more precisely that $\chi_L(R) = O(R)$ as $R \to 0$.