Project I: Partial Wave Solutions of Quantum Scattering

Let us consider proton-neutron scattering, i.e. we have a two-body system consisting of a neutron and a proton, and we want to calculate a selected set of phase shifts for a simplified proton-neutron potential, in which the spin of the proton and neutron are neglected. Such a simple representation of a neutron-proton potential is the Malfliet-Tjon potential given below.

V_A [MeV fm]	μ_A [fm^{-1}]	V_R[MeV fm]	μ_R [fm^{-1}]
626.8932	1.550	1438.7228	3.11

Table 1: The parameters for the Malfliet-Tjon type potential of our calculation [2]. As conversion factor we use units such that $\hbar c$=197.3286 MeV fm.

In order to get started and check you differential equation:

1. Bound State Calculation:

The neutron-proton system supports one bound state, the deuteron.

1. Plot the potential.
2. Use your bound state code and find the s-wave bound state of the deuteron as given by this potential. Calculate this bound state with a precision of 4 significant figures, and discuss the numerical test you carried out to achieve this accuracy. Pay attention for $r \to 0$. You may have to loosen the conditions of your Wood-Saxon code for small values of r.
   Hint: The binding energy of the deuteron is of the order of -2 MeV. Use for the neutron and proton masses their average, 938.926 MeV.
3. Calculate and plot the normalized ground state wave function.

The standard partial wave decomposition of a scattering wave function is

$$\Psi(\mathbf{r}) = \sum_{l=0}^{\infty}(2l+1)i^l e^{i\delta_l}\frac{u_l(r)}{kr}P_l(\cos\theta), \tag{1}$$

where k is related to the c.m. energy as $E_{c.m.} = k^2/2\mu$, which is positive. When this expansion is substituted into the Schrödinger equation, the radial wave functions $u_l(r)$ are found to satisfy the radial equations

$$\left[-\frac{\hbar^2}{2\mu}\frac{d^2}{dr^2} + V(r) + \frac{l(l+1)\hbar^2}{2\mu r^2} - E\right]u_l(r) = 0. \tag{2}$$

Although this is the same equation as the one satisfied by a bound state wave function, the boundary conditions are different. In particular, $u_l(r)$ vanishes at the origin, but it has the large-r asymptotic behavior

$$u_l(r) \to kr\left[\cos\delta_l j_l(kr) - \sin\delta_l n_l(kr)\right], \tag{3}$$

where $j_l(kr)$ and $n_l(kr)$ are the regular and irregular spherical Bessel functions of order l.

The scattering amplitude $f_E(\theta)$ is related to the phase shift δ_l by

$$f_E(\theta) = \frac{1}{k}\sum_{l=0}^{\infty}(2l+1)e^{i\delta_l}\sin\delta_l P_l(\cos\theta), \tag{4}$$

and the total cross section is given by

$$\sigma(E) = \frac{4\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)\sin^2\delta_l. \tag{5}$$

The derivation of those equations can be found in your lecture notes of Phys 612, and under *www.phy.ohiou.edu/˜ elster/lectures/qm1_11.pdf*, specifically Section 11.5.

Considering the boundary conditions for scattering, and setting the logarithmic derivative

$$\gamma := \frac{1}{u_l(r)}\frac{d}{dr}u_l(r) \tag{6}$$

equal to the one for the free solution at a fixed $r \equiv r_{match}$ leads to a condition for the phase shift which is then given by

$$\tan\delta_l = \frac{\gamma J_l(kr) - J_l'(kr)}{\gamma N_l(kr) - N_l'(kr)}, \tag{7}$$

where $J_l(kr) \equiv r j_l(kr)$ and $N_l(kr) \equiv r n_l(kr)$.

For your calculation of proton-neutron scattering for c.m. energies $E = 10$ MeV to $E = 150$ MeV with the Malfliet-Tjon Potential calculate the following:

2. Scattering Calculation:

1. Calculate the phase shifts δ_l for $l = 0, 1, 2, 3, 4$ and plot them as function of E. Plot the phase shifts in degrees.
   Hint: It will be sufficient for a plot to use an energy step of 5 MeV.
2. Convince yourself (and me) that your calculated phase shifts are accurate within 3 significant figures and discuss the tests you performed to come to the required accuracy.
3. Plot the scattering amplitude as $f_E(\theta)$ as function of θ for the following energies: 10 MeV, 50 MeV, 100 MeV, 150 MeV. Comment on how many partial waves are needed to have a converged result for $f_E(\theta)$ for the different energies. Can you relate the shape of $f_E(\theta)$ to the properties of the Legendre polynomials?
   Hint: Make sure you have the correct units for $f_E(\theta)$.

4. Calculate the cross section as function of energy from $E = 10$ MeV to $E = 150$ MeV.

Remember, all plots **must** have units as part of the axes in order to receive full credit. And if differences in calculations are not visible in a figure they must be included as table.

Good Luck!

References:

[1] R.A. Malfliet and J.A. Tjon, Nucl. Phys. **A127**, 161 (1969).

[2] H. Liu, Ch. Elster, W. Glöckle, Phys. Rev. **C72**, 054003 (2005).

A Malfliet-Tjon type potential [1,2] has the form

$$V_{MT}(r) = V_R \frac{e^{-\mu_R r}}{r} - V_A \frac{e^{-\mu_A r}}{r} \tag{8}$$