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INSTITUTE OF NUCLEAR & PARTICLE PHYSICS

Single Channel Scattering

With charged particles

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Lecture 4

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U.S. DEPARTMENT OF ENERGY

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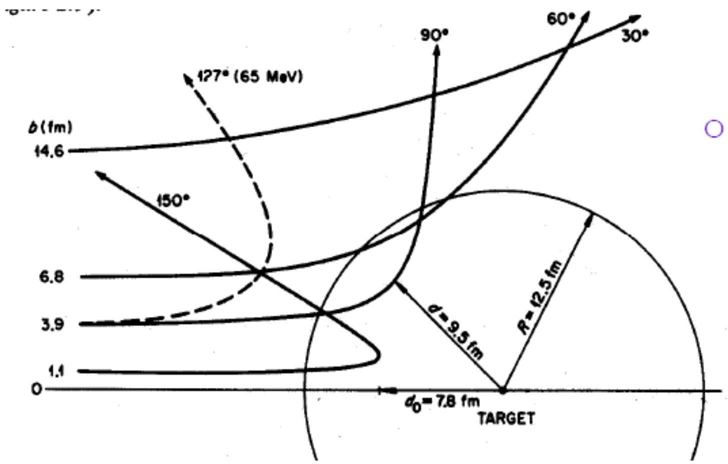
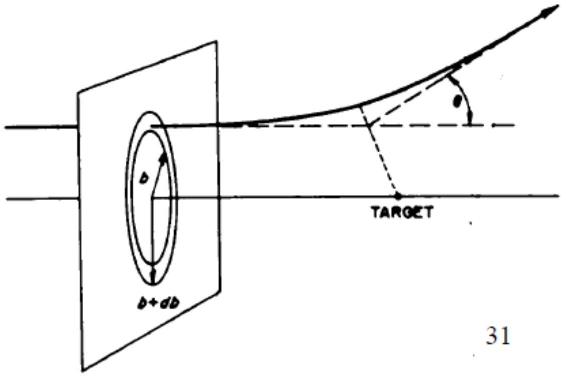
Classical Coulomb Scattering



- Coulomb trajectories are hyperbolas

$$\tan\frac{\theta}{2} = \frac{\eta}{bk}$$



Rutherford formula for Coulomb cross section

$$\sigma(\theta) \equiv \frac{b(\theta)}{\sin\theta}\frac{\mathrm{d}b}{\mathrm{d}\theta} = \frac{\eta^2}{4k^2\sin^4(\theta/2)}$$

Coulomb functions

$$\left[\frac{d^2}{d\rho^2} - \frac{L(L+1)}{\rho^2} - \frac{2\eta}{\rho} + 1\right] X_L(\eta,\rho) = 0$$

Schrödinger eq. with Coulomb potential

Solution:

$$F_L(\eta,\rho) = C_L(\eta)\rho^{L+1} e^{\mp i\rho}\ {}_1F_1(L+1 \mp i\eta; 2L+2; \pm 2i\rho)$$

$$C_L(\eta) = \frac{2^L e^{-\pi\eta/2} |\Gamma(1+L+i\eta)|}{(2L+1)!}$$

$${}_1F_1(a;b;z) = 1 + \frac{a}{b}\frac{z}{1!} + \frac{a(a+1)}{b(b+1)}\frac{z^2}{2!} + \frac{a(a+1)(a+2)}{b(b+1)(b+2)}\frac{z^3}{3!} + \cdots .$$

$$\begin{aligned} H_L^{\pm}(\eta,\rho) &= G_L(\eta,\rho) \pm iF_L(\eta,\rho) \\ &= e^{\pm i\Theta} (\mp 2i\rho)^{1+L\pm i\eta} U(1+L \pm i\eta, 2L+2, \mp 2i\rho) \end{aligned}$$

$$\Theta \equiv \rho - L\pi/2 + \sigma_L(\eta) - \eta \ln(2\rho)$$

$$\sigma_L(\eta) = \arg\Gamma(1+L+i\eta)$$

Coulomb functions

Behavior near the origin:

$$F_L(\eta,\rho) \sim C_L(\eta)\rho^{L+1}, \quad G_L(\eta,\rho) \sim \left[(2L+1)C_L(\eta)\,\rho^L\right]^{-1}$$

$$C_0(\eta) = \sqrt{\frac{2\pi\eta}{e^{2\pi\eta}-1}} \quad \text{and} \quad C_L(\eta) = \frac{\sqrt{L^2+\eta^2}}{L(2L+1)}C_{L-1}(\eta)$$

Behavior at large distances:

$$F_L(\eta,\rho) \sim \sin\Theta, \quad G_L(\eta,\rho) \sim \cos\Theta, \quad \text{and} \quad H_L^{\pm}(\eta,\rho) \sim e^{\pm i\Theta}$$

$$\Theta \equiv \rho - L\pi/2 + \sigma_L(\eta) - \eta\ln(2\rho)$$

Coulomb scattering in partial waves

Schrödinger equation with Coulomb potential has exact solution

Reminder: $V_c(R) = Z_1 Z_2 e^2 / R$

$$\psi_c(\mathbf{k},\mathbf{R}) = \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{R}}\, \mathrm{e}^{-\pi\eta/2}\, \Gamma(1+\mathrm{i}\eta)\, {}_1F_1(-\mathrm{i}\eta; 1; \mathrm{i}(kR - \mathbf{k}\cdot\mathbf{R}))$$

- generalize the partial wave form of the plane wave

$$\psi_c(k\hat{\mathbf{z}},\mathbf{R}) = \sum_{L=0}^{\infty} (2L+1)\mathrm{i}^L P_L(\cos\theta)\frac{1}{kR} F_L(\eta, kR)$$

- asymptotic form of the scattering wavefunction

$$\psi_c(k\hat{\mathbf{z}},\mathbf{R}) \overset{R-Z\to\infty}{\to} \mathrm{e}^{\mathrm{i}[kz+\eta\ln k(R-z)]} + f_c(\theta)\frac{\mathrm{e}^{\mathrm{i}[kR-\eta\ln 2kR]}}{R}$$

Coulomb scattering amplitude

- formally can be written in partial wave expansion

$$f_c(\theta) = \frac{1}{2ik}\sum_{L=0}^{\infty}(2L+1)P_L(\cos\theta)(e^{2i\sigma_L(\eta)} - 1)$$

Series does not converge

- without partial wave expansion one can derive the scattering amplitude

$$f_c(\theta) = -\frac{\eta}{2k\sin^2(\theta/2)}\exp\left[-i\eta\ln(\sin^2(\theta/2)) + 2i\sigma_0(\eta)\right]$$

Cross section for point Coulomb (Rutherford cross section)

$$\sigma_{\text{Ruth}}(\theta) = |f_c(\theta)|^2 = \frac{\eta^2}{4k^2\sin^4(\theta/2)}$$

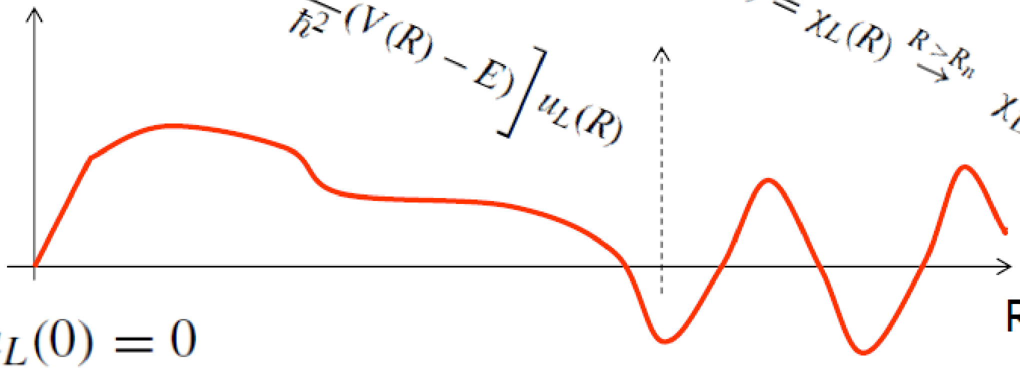
Scattering problem Coulomb + short range force

- numerical solution is proportional to true solution $\chi_L(R) = Bu_L(R)$

$$u_L''(R) = \left[\frac{L(L+1)}{R^2} + \frac{2\mu}{\hbar^2}(V(R)-E)\right]u_L(R)$$

Coulomb+nuclear

$$Bu_L(R) = \chi_L(R) \xrightarrow{R>R_n} \chi_L^{ext}(R) = \frac{i}{2}[H_L^-(\eta,kR) - \mathbf{S}_L^n H_L^+(\eta,kR)]$$



$$u_L(0) = 0$$
$$u'_L(0) \neq 0$$

- generalized asymptotic form defines the nuclear S-matrix

$$\chi_L^{\text{ext}}(R) = \frac{\mathrm{i}}{2}[H_L^-(\eta, kR) - \mathbf{S}_L^n H_L^+(\eta, kR)]$$

- can be written in terms of the nuclear phase shift

$$\mathbf{S}_L^n = \mathrm{e}^{2\mathrm{i}\delta_L^n}$$

$$\chi_L^{\text{ext}}(R) = \mathrm{e}^{\mathrm{i}\delta_L^n}\left[\cos\delta_L^n\, F_L(\eta, kR) + \sin\delta_L^n\, G_L(\eta, kR)\right]$$

- combined phase shift from Coulomb and nuclear

$$\delta_L = \sigma_L(\eta) + \delta_L^n$$

Phase shifts and cross section:

$$\delta_L = \sigma_L(\eta) + \delta_L^n$$ Coulomb + nuclear phase shifts

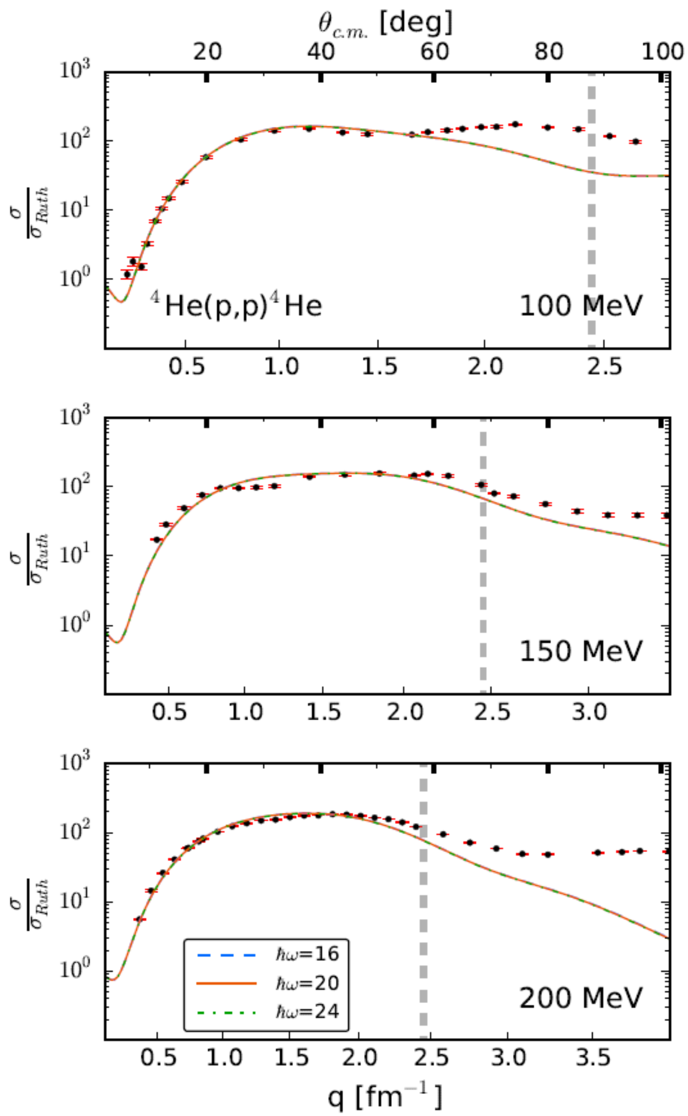
$$e^{2i\delta_L} - 1 = (e^{2i\sigma_L(\eta)} - 1) + e^{2i\sigma_L(\eta)}(e^{2i\delta_L^n} - 1)$$

$$f_{nc}(\theta) = f_c(\theta) + f_n(\theta)$$

$$f_n(\theta) = \frac{1}{2ik}\sum_{L=0}^{\infty}(2L+1)P_L(\cos\theta)e^{2i\sigma_L(\eta)}(\mathbf{S}_L^n - 1)$$

$$\sigma_{nc}(\theta) = |f_c(\theta) + f_n(\theta)|^2 \equiv |f_{nc}(\theta)|^2$$

Often shown: $\sigma/\sigma_{\text{Ruth}} \equiv \sigma_{nc}(\theta)/\sigma_{\text{Ruth}}(\theta)$



Burrows, Elster, Weppner, Launey, Maris, Nogga, Popa Phys. Rev. **C99**, 044603 (2019).